plex algorithm; Linear programming; Lemke method; Integer linear complementary problem; LCP: Pardalos–Rosen mixed integer formulation; Order complementarity; Generalized nonlinear complementarity problem; Topological methods in complementarity theory.

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ESTIMATING DATA FOR MULTICRITERIA DECISION MAKING PROBLEMS: OPTIMIZATION TECHNIQUES by Triantaphyllou et al.

One of the most crucial steps in many multicriteria decision making methods (MCDM) is the accurate estimation of the pertinent data [18]. Very often these data cannot be known in terms of absolute values. For instance, what is the worth of the ith alternative in terms of a political impact criterion? Although information about questions like the previous one is vital in making the correct decision, it is very difficult, if not impossible, to quantify it correctly. Therefore, many decision making methods attempt to determine the *relative* importance, or weight, of the alternatives in terms of each criterion involved in a given decision making problem.

Consider the case of having a single decision criterion and a set of n alternatives, denoted as A_i (for $i = 1, \dots, n$). The decision maker wants to determine the relative performance of these alternatives in terms of a single criterion. An approach based on *pairwise comparisons* which was proposed by T.L. Saaty [11], and [12] has long attracted the interest of many researchers, because both of its easy applicability and interesting mathematical properties. Pairwise comparisons are used

* For the reference see the last page.

to determine the relative importance of each alternative in terms of each criterion.

In that approach the decision maker has to express his/her opinion about the value of one single pairwise comparison at a time. Usually, the decision maker has to choose his/her answer among 10–17 discrete choices. Each choice is a linguistic phrase. Some examples of such linguistic phrases when two concepts, **A** and **B** are considered might be: '**A** is more important than **B**', or '**A** is of the same importance as **B**', or '**A** is a little more important than **B**', and so on. When one focuses directly on the *data elicitation* issue one may use linguistic statements such as 'How much more does alternative **A** belong to the set **S** than alternative **B**'?

The main problem with the pairwise comparisons is how to quantify the *linguistic choices* selected by the decision maker during the evaluation of the pairwise comparisons. All the methods which use the pairwise comparisons approach eventually express the qualitative answers of a decision maker into some numbers.

Pairwise comparisons are quantified by using a *scale*. Such a scale is nothing but an one-to-one mapping between the set of discrete linguistic choices available to the decision maker and a discrete set of numbers which represent the importance, or weight, of the previous linguistic choices. There are two major approaches in developing such scales. The first approach is based on the *linear scale* proposed by Saaty [12] as part of the *analytic hierarchy process* (AHP). The second approach was proposed by F. Lootsma [8], [9], [10] and determines *exponential scales*. Both approaches depart from some psychological theories and develop the numbers to be used based on these psychological theories. For an extensive study of the scale issue, see [18] and [19].

In this article we examine three problems related to the use of pairwise comparisons for data elicitation in MCDM. The first problem is how to combine the $n(n-1)/2$ comparisons needed to compare n entities (alternatives or *criteria*) under a given goal and extract their relative preferences. This subject was extensively studied in [21] and it is briefly discussed in the second section. The second problem in this article is how to estimate *missing comparisons*. The third problem is how to select the order for eliciting the comparisons and determine whether all comparisons are needed. These problems are examined in detail in the following sections.

Extraction of Relative Priorities from Complete Pairwise Matrices. Let $A_1, \ldots, A_n$ be n alternatives (or criteria or, in general, concepts) to be compared. We are interested in evaluating the relative preference values of the above concepts. Saaty [11], [12], [14] proposed to use a matrix A of rational numbers taken from the set $\{1/9, 1/8, 1/7, \ldots, 1, \ldots, 9\}$. Each entry of the above matrix A represents a *pairwise judgment*. Specifically, the entry a_{ij} denotes the number that estimates the relative preference of element A_i when it is compared with element A_j. Obviously, $a_{ij} = 1/a_{ji}$ and $a_{ii} = 1$. That is, the matrix is reciprocal.

The Eigenvalue Approach. Let us first examine the case in which it is possible to have perfect values a_{ij}. In this case it is $a_{ij} = W_i/W_j$ (W_s denotes the actual value of element s) and the previous reciprocal matrix A is *consistent*. That is:

$$a_{ij} = a_{ik} \times a_{kj} \quad \text{for } i, j, k = 1, \ldots, n, \tag{1}$$

where n is the number of elements in the comparison set. It can be proved [12] that the matrix A has rank 1 with n to be its nonzero eigenvalue. Thus, we have:

$$Ax = nx, \tag{2}$$

where x is an eigenvector. From the fact that $a_{ij} = W_i/W_j$, the following are obtained:

$$\sum_{j=1}^{n} a_{ij} W_j = \sum_{j=1}^{n} W_i = nW_i, \quad i = 1, \ldots, n, \tag{3}$$

or

$$AW = nW. \tag{4}$$

Equation (4) states that n is an eigenvalue of [...] with W being a corresponding eigenvector. [...] same equation also states that in the *perfectly* [...]*sistent case* (i.e., when $a_{ij} = a_{ik} \times a_{kj}$ for all po[...]ble triplets), the vector W, with the relative [...]

ences of the elements $A_1, \ldots, A_n$, is the principal right eigenvector (after normalization) of A.

In the nonconsistent case (which is the most common) the pairwise comparisons are not perfect, that is, the entry a_{ij} might deviate from the real ratio W_i/W_j (i.e., from the ratio of the real relative preference values W_i and W_j). In this case, the previous expression (1) does not hold for all possible combinations. Now the new matrix A can be considered as a perturbation of the previous consistent case. When the entries a_{ij} change slightly, then the eigenvalues change in a similar fashion [12]. Moreover, the maximum eigenvalue is close to n (actually greater than n) while the remaining eigenvalues are close to zero. Thus, in order to find the relative preferences in the nonconsistent cases, one should find an eigenvector that corresponds to the maximum eigenvalue $\lambda_{\max}$. That is to say, to find the principal right eigenvector W that satisfies:

$$AW = \lambda_{\max} W \quad \text{where } \lambda_{\max} = n.$$

Saaty estimates the principal right eigenvector W by multiplying the entries in each row of A together and taking the nth root (n being the number of the elements in the comparison set). Since we desire to have values that add up to 1, we normalize the previously found vector by the sum of the above values. If we want to have the element with the highest value to have a relative preference value equal to 1, we divide the previously found vector by the highest value.

Under the assumption of *total consistency*, if the judgments are gamma distributed (something that Saaty claims to be the case), the principal right eigenvector of the resultant reciprocal matrix A is Dirichlet distributed. If the assumption of total consistency is relaxed, then L.G. Vargas [23] proved that the hypothesis that the principal right eigenvector follows a Dirichlet distribution is accepted if the consistency ratio is 10% or less.

The *consistency ratio* (CR) is obtained by first estimating $\lambda_{\max}$. Saaty estimates $\lambda_{\max}$ by adding the columns of matrix A and then multiplying the resulting vector with the vector W. Then, he uses what he calls the *consistency index* (CI) of the matrix A. He defined CI as follows:

$$\text{CI} = \frac{\lambda_{\max} - n}{n - 1}.$$

Then, the consistency ratio CR is obtained by dividing the CI by the random consistency index (RCI) as given in table 1. Each RCI is an average random consistency index derived from a sample of size 500 of randomly generated reciprocal matrices with entries from the set $\{1/9, 1/8, 1/7, \ldots, 1, \ldots, 9\}$ to see if its CI is 10% or less. If the previous approach yields a CR greater than 10%, then a reexamination of the pairwise judgments is recommended until a CR less than or equal to 10% is achieved.

Optimization Approaches. A.T.W. Chu, R.E. Kalaba and K. Spingarn [2] claimed that given the data a_{ij}, the values W_i to be estimated are desired to have the property:

$$a_{ij} \approx \frac{W_i}{W_j}. \tag{5}$$

This is reasonable since a_{ij} is meant to be the estimation of the ratio W_i/W_j. Then, in order to get the estimates for the W_i given the data a_{ij}, they proposed the following constrained optimization problem:

$$\begin{cases} \min & S = \sum_{i=j}^{n} \sum_{j=i}^{n} (a_{ij} w_j - w_i)^2, \\ \text{s.t.} & \sum_{i=j}^{n} W_i = 1, \\ & W_i > 0 \quad \text{for } i = 1, \ldots, n. \end{cases} \tag{6}$$

They also provide an alternative expression S_1 that is more difficult to solve numerically. That is,

$$S_1 = \sum_{i=j}^{n} \sum_{j=i}^{n} (a_{ij} - W_j/W_i)^2 . \tag{7}$$

In [3] a variation of the above *least squares* formulation is proposed. For the case of only one decision maker it recommends the following models:

$$\log a_{ij} = \log W_i - \log W_j + \psi_2(W_i, W_j)\varepsilon_{ij}, \tag{8}$$

$$a_i j = \frac{W_i}{W_j} + \psi_2(W_i, W_j)\varepsilon_{ij}, \tag{9}$$

where W_i and W_j are the true (and hence unknown) relative preferences; $\psi_1(X, Z)$ and $\psi_2(X, Z)$ are given positive functions (where

n	1	2	3	4	5	6	7	8	9
RCI	0	0	0.58	0.90	1.12	1.24	1.32	1.41	1.45

Table 1: RCI values for sets of different order n [12].

$X, Z > 0$). The random errors ε_{ij} are assumed independent with zero mean and unit variance. Using these two assumptions one is able to calculate the variance of each individual estimated relative preference. However, is fails to give a way of selecting the appropriate positive functions. In the second example, presented later, a sample problem which originates in [11] and later in [3] is solved for different functions ψ_1, ψ_2 using this method.

Considering the Human Rationality Factor. According to the *human rationality assumption* [21] the decision maker is a rational person. Rational persons are defined here as individuals who try to minimize their regret [15], to minimize losses, or to maximize profit [24]. In the relative preference evaluation problem, *minimization of regret*, losses, or maximization of profit could be interpreted as the effort of the decision maker to minimize the errors involved in the pairwise comparisons.

As it is stated in previous paragraphs, in the inconsistent case the entry a_{ij} of the matrix A is an estimation of the real ratio W_i/W_j. Since it is an estimation, the following is true:

$$a_{ij} = \left(\frac{W_i}{W_j}\right) d_{ij}, \quad i, j = 1, \ldots, n. \tag{10}$$

In the above relation d_{ij} denotes the deviation of a_{ij} from being an accurate judgment. Obviously, if $d_{ij} = 1$, then the a_{ij} was perfectly estimated. From the previous formulation we conclude that the errors involved in these pairwise comparisons are given by:

$$\varepsilon_{ij} = d_{ij} - 1.00,$$

or after using (10), above:

$$\varepsilon_{ij} = a_{ij}\left(\frac{W_j}{W_i}\right) - 1.00. \tag{11}$$

When a comparison set contains n elements, then Saaty's method requires the estimation of the following $n(n-1)/2$ pairwise comparisons:

$$\frac{W_2}{W_1}, \ldots, \frac{W_n}{W_1}, \tag{12}$$

$$\frac{W_3}{W_2}, \ldots, \frac{W_n}{W_2},$$

$$\vdots$$

$$\frac{W_{n-1}}{W_n}.$$

The corresponding $n(n-1)/2$ errors are (after using relations (11) and (12)):

$$\varepsilon_{ij} = a_{ij}\left(\frac{W_j}{W_i}\right) - 1.00, \tag{13}$$

$$i, j = 1, \ldots, n, \text{ and } j > 1.$$

Since the W_i are relative preferences that add up to 1, the following relation (14) should also be satisfied:

$$\sum_{i=1}^{n} W_i = 1.00. \tag{14}$$

Apparently, since the W_i represent relative preferences we also have:

$$W_i > 0, \quad i = 1, \ldots, n. \tag{15}$$

Relations (13) and (14), when the data are consistent (i.e., all the errors are equal to zero), can be written as follows:

$$BW = b. \tag{16}$$

The vector b has zero entries everywhere except the last one that is equal to 1, and the matrix B has the following form (blank entries represent zeros):

$$B = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ -1 & a_{1,2} & & & & 1 \\ -1 & & a_{1,3} & & & 2 \\ \vdots & & & \ddots & & \vdots \\ -1 & & & & a_{1,n} & n-1 \\ & -1 & a_{2,3} & & & 1 \\ & \vdots & & \ddots & & \vdots \\ & -1 & & & a_{2,n} & n-2 \\ & & \ddots & & & \vdots \\ & & & & a_{n-1,n} & 1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

The *error minimization* issue is interpreted in many cases (regression analysis, linear least squares problem) as the minimization of the *sum of squares* of the residual vector: $r = b - BW$ [16]. In terms of formulation (15) this means that in a real life situation (i.e., when errors are not zero any more) the real intention of the decision maker is to minimize the expression:

$$f^2(x) = \|b - BW\|, \tag{17}$$

which, apparently, expresses a typical linear least squares problem.

If we use the notation described previously, then the quantity (6) which is minimized in [2] becomes:

$$S = \sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij}W_j - W_i)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}(\varepsilon_{ij}W_i)^2$$

and the alternative expression (7) becomes:

$$S_1 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(a_{ij}\frac{W_j}{W_i}\right)^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\varepsilon_{ij}\frac{W_i}{W_j}\right)^2.$$

Clearly, both expressions are too complicated to reflect, in a reasonable way, the intentions of the decision maker.

The models proposed in [3] are closer to the one developed under the human rationality assumption. The only difference is that instead of the relations:

$$\log a_{ij} = \log w_i - \log W_j + \psi_1(W_i, W_j)\varepsilon_{ij}$$

and

$$a_{ij} = \frac{W_i}{W_j} + \psi_2(W_i, W_j)\varepsilon_ij,$$

the following simpler expression is used:

$$a_{ij} = \frac{W_i}{W_j}d_{ij}, \tag{18}$$

or

$$a_{ij} = \frac{W_i}{W_j} \times (\varepsilon_{ij} + 1.00).$$

However, as the second example illustrates, the performance of this method is greatly dependent on the selection of the $\psi_1(X, Z)$ or $\psi_2(X, Z)$ functions. Now, however, these functions are further modified by (17).

EXAMPLE 1 Let us assume that the following is the matrix with the pairwise comparisons for a set of four elements:

$$A = \begin{bmatrix} 1 & 2/1 & 1/5 & 1/9 \\ 1/2 & 1 & 1/8 & 1/9 \\ 5/1 & 8/1 & 1 & 1/4 \\ 9/1 & 9/1 & 4/1 & 1 \end{bmatrix}.$$

Using the methods presented in previous sections we can see that

$$\lambda_{\max} = 4.226;$$

$$\mathrm{CI} = \frac{4.226 - 4}{4 - 1} = 0.053,$$

$$\mathrm{CR} = \frac{\mathrm{CI}}{0.90} = 0.0837 < 0.10.$$

The formulation (15) that corresponds to this example is as follows:

$$\begin{bmatrix} -1 & 2/1 & 0.0 & 0 \\ -1 & 0.0 & 1/5 & 0 \\ 1 & 0.0 & 0 & 1/9 \\ 0.0 & -1 & 1/8 & 0 \\ 0.0 & -1 & 0 & 1/9 \\ 0.0 & 0.0 & -1 & 1/4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1.0 \end{bmatrix}.$$

The vector V that solves the above least squares problem is calculated to be:

$$V = (0.065841\ 0.039398\ 0.186926\ 0.704808).$$

Hence, the sum of squares of the residual vector components is 0.003030. The average squared residual for this problem is $0.003030/((4(4-1)/2) + 1 = 0.000433$; that is, the average residual is $\sqrt{0.000433} = 0.020806$. □

EXAMPLE 2 The second example uses the same data used originally in [11], and later in [2] and [3]. These data are presented in Table 2.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1)	1	4	9	6	6	5	5
(2)	1/4	1	7	5	5	3	5
(3)	1/9	1/7	1	1/5	1/5	1/7	1/5
(4)	1/6	1/5	5	1	1	1/3	1/3
(5)	1/6	1/5	5	1	1	1/3	1/3
(6)	1/5	1/3	7	3	3	1	2
(7)	1/5	1/4	5	3	3	1/2	1

Table 2: Data for the second example.

Table 3 presents a summary of the results (as found in the corresponding references) when the methods described in the subsections above are used. The *power method* for deriving the eigenvector was applied as presented in [7]. In the last row of Table 2 are the results obtained by using the least square method under the human rationality assumption (HR).

As it is shown in the last column of Table 3, the performance of each method is very different as far the mean residual is concerned. The results also illustrate how critical is the role of the functions $\psi_1(X, Z)$ and $\psi_2(X, Z)$ in the method of [3]. The mean residual obtained by using the least squares method under the human rationality assumption is the smallest one by 16%. □

Matrices with Missing Comparisons. For one to evaluate n concepts, normally all the required $n(n-1)/2$ pairwise comparisons are needed. However, for large numbers of concepts to be compared, the decision maker may become quite bored, tired and inattentive with assigning the values to the comparisons as time is going on, which may easily lead to erroneous judgments. Moreover, the time spent to elicit all the comparisons for a judgment matrix may be unaffordable. Also the decision maker may not be sure about the values of some comparisons and thus may not want to make a direct evaluation of them. In cases like the previous ones, the decision maker may wish to stop the process and then try to derive the relative preferences from an incomplete pairwise comparison (judgment) matrix.

Given an incomplete pairwise comparison matrix, there are two central and closely interrelated problems. The first problem is how to estimate the missing comparisons. The second problem is which comparison to evaluate next. In other words, if the decision maker wishes to estimate a few extra comparisons (from the remaining undetermined ones) how should the next comparison be selected? Should it be selected randomly or according to some rule (to be determined)? Next, we study the first of these two closely related problems.

Estimating Missing Comparisons.

Using Connecting Paths. Suppose that $X_{i,j}$ is a missing comparison to be estimated. Next, also assume that there are two known comparisons $a_{i,k}$ and $a_{k,j}$ for some index k. In the perfectly consistent case the following relationship should be true:

$$X_{i,j} = a_{i,k} \times a_{k,j}.$$

In the more general inconsistent case, the $X_{i,j}$ value can be approximated by the product $a_{i,k} \times a_{k,j}$. In [5], and [6] the pair $a_{i,k}$ and $a_{k,j}$ is called an *elementary connecting path* connecting the missing comparison $X_{i,j}$. Obviously, given a missing comparison, more than one such connecting path may exist (i.e., if there are more than one k indexes which satisfy the above relationship). Moreover, it is also possible to have connecting paths comprised by more than two known comparisons (i.e., paths of size larger than 2). The general structure of a connecting path of size r, denoted as CP_r, has the following form:

$$\mathrm{CP}_r: \quad X_{i,j} = a_{i,k1} \times a_{k1,k2} \times \cdots \times a_{kr,j},$$

for $i, j, k1, \ldots, kr = 1, \ldots, n$, $1 \leq r \leq n-2$.

According to P.T. Harker [5], [6] the value of the missing comparison $X_{i,j}$ should be equal to the geometric mean of all connecting paths related to this missing comparison. That is, the following should be true:

$$X_{ij} = \sqrt[q]{\prod_{r=1}^{q} \mathrm{CP}_r}.$$

In the previous expression it is assumed that there are q such connecting paths. For the above reasons, this method is known as the *geometric mean method* for estimating missing comparisons.

A method alternative to the geometric means method is to express the missing comparisons in terms of the arithmetic averages of all related connecting paths and some error terms. In this way, one can also introduce error terms on consistency relations which are defined on pairs of missing comparisons (for more details, please see [1]). A natural objective then, could be to minimize the sum of the absolute terms of all these error terms (which can be of any sign). That is, the above consideration leads to the formulation of a linear programming (LP) problem. A similar approach is presented in [17] (in which the path problem does not occur).

However, there is a serious drawback with any method which attempts to use connecting paths. The number of connecting paths may be astronomically large, rendering any such method computa-

method used	elements in set (1)	(2)	(3)	(4)	(5)	(6)	(7)	Ave. residual
Saaty eigenvector method	0.429	0.231	0.021	0.053	0.053	0.119	0.095	0.134
Power method eigenvector	0.427	0.230	0.021	0.052	0.052	0.123	0.094	0.135
Chu's method	0.487	0.175	0.030	0.059	0.059	0.104	0.085	0.097
Federov model 1 with $\psi_1 = 1$	0.422	0.232	0.021	0.052	0.052	0.127	0.094	0.138
Federov Model 2 with $\psi_2 = 1$	0.386	0.287	0.042	0.061	0.061	0.088	0.075	0.161
Federov Model 2 with $\psi_2 = \lvert W_i - W_j \rvert$	0.383	0.262	0.032	0.059	0.059	0.122	0.083	0.152
Federov Model 2 with $\psi_2 = W_i/W_j$	0.047	0.229	0.021	0.051	0.051	0.120	0.081	0.130
Least squares method under the HR assumption	0.408	0.147	0.037	0.054	0.054	0.080	0.066	0.082

Table 3: Comparison of the relative preferences for the data in Table 2.

tionally intractable. For instance, for a comparison matrix of dimension of six, the number of possible connecting paths to be considered might be equal to 64, while in a case of dimension equal to ten, the number of paths may become equal to 109,600. As a result, some alternative approaches have been developed. The revised geometric means method (or RGM) method and a least squares formulation are two such methods and are discussed next.

Revised Geometric Mean Method (RGM). An alternative approach to the use of connecting paths, is to convert the incomplete judgement matrix into a transformed matrix and then determine its principal right eigenvector. This was proposed by Harker [4] and it is best illustrated by means of an example.

Suppose that the following is an incomplete judgement matrix of order 3 (taken from [4]).

$$A_0 = \begin{bmatrix} 1 & 2 & - \\ 1/2 & 1 & 2 \\ - & 1/2 & 1 \end{bmatrix}.$$

One can replace the missing elements (denoted by $-$) by the corresponding ratios of weights. Therefore, the previous matrix becomes:

$$A_1 = \begin{bmatrix} 1 & 2 & w_1/w_3 \\ 1/2 & 1 & 2 \\ w_3/w_1 & 1/2 & 1 \end{bmatrix}.$$

That is, the missing comparison $X_{1,3}$ was replaced by the ratio w_1/w_3 (similar for the reciprocal entry $X_{3,1}$). Next observe that the product $A_1 W$ is equal to:

$$A_1 W = \begin{bmatrix} 1 & 2 & w_1/w_3 \\ 1/2 & 1 & 2 \\ w_3/w_1 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2w_1 + 2w_2 \\ w_1/2 + w_2 + 2w_3 \\ w_2/2 + 2w_3 \end{bmatrix}$$

The same result can also be obtained if one considers the matrix C, given as follows:

$$C = \begin{bmatrix} 2 & 2 & 0 \\ 1/2 & 1 & 2 \\ 0 & 1/2 & 2 \end{bmatrix},$$

that is, matrix C satisfies the relationship

$$A_1 W = CW.$$

Therefore, the desired relative preferences (i.e., the entries of vector W) can be determined as the principal right eigenvector of the new matrix C. This is true because:

$$A_1 W = CW = \lambda W.$$

In general, the entries of matrix C can be determined from the entries of an incomplete judgement matrix A_0 as follows (where $c_{i,j}$ and $a_{i,j}$ are the elements of the matrices C and A_0, respectively):

$$c_{i,i} = 1 + m_i$$

and for $i \neq j$:

$$c_{i,j} = \begin{cases} a_{i,j} & \text{if } a_{i,j} \text{ is a positive number,} \\ 0 & \text{otherwise,} \end{cases}$$

where m_i is the number of unanswered questions in the ith row of the incomplete comparison matrix.

Next, the elements of the W vector can be determined by using one of the methods presented in the second section.

Least Squares Formulation. This formulation is a natural extension of the formulation discussed earlier in the section on the HR factor. The only difference is that in relations (12) one should only consider known comparisons. This, as a result, implies that the new matrix B (as defined earlier) should not have rows which would correspond to missing comparisons. Finally, observe that in order to solve the least squares problem given as (16), one has to calculate the vector W as follows:

$$W = (B^\top B)^{-1} B^\top b,$$

where $B^\top$ stands for the transpose of B.

In [1] the revised geometric means and the previous least squares method were tested on random problems. First, a complete judgment matrix was determined. These matrices, in general, were slightly inconsistent. They were derived according to the procedures used in [22], [20], and [19]. Then, some comparisons were randomly removed and set as missing. Then, the previous two methods were applied on the incomplete judgment matrix and the missing comparisons were estimated. The estimated matrix was used to derive a ranking of the compared entities. This ranking was compared with the ranking derived when the original complete judgment matrix is used. In these computational experiments it was found that the two estimation methods for missing comparisons performed almost in a similar manner. This manner was different for matrices of different order and various percentages of missing comparisons. More details on these issues can be found in [1].

Determining the Comparison to Elicit Next. Suppose that the decision maker has determined some of the $n(n-1)/2$ comparisons when a set of n entities is considered for extracting relative preferences. Next assume that the decision maker wishes to proceed with only a few additional comparisons and not determine the entire judgment matrix. The question we examine at this point is which ones the additional comparisons should be. To be more specific, the question we consider is best stated as follows: Given an incomplete judgment matrix, and the option to elicit just some additional comparisons, then which one should be the comparison to elicit next?

One obvious approach is to select the next comparison just randomly among the missing ones. This problem was examined by Harker in [5] and [6]. Harker focused his attention on how to determine which comparison, among the missing ones, is the most *critical* one. He determined as the most critical one, to be the comparison which would have the largest impact (when the appropriate derivatives are considered) on the vector W.

He observed that the largest absolute gradient (i.e., the largest partial derivative) means that a unit change of the specific missing comparison brings out the biggest change on the vector W. Therefore, he asserted, that the missing comparison related to the largest absolute gradient should be the most critical one and therefore, the one to evaluate next. Then, the following formula calculating the largest absolute gradient can be used to choose the most critical comparison index (i,j):

$$(i,j) = \arg \max_{(k,l)\epsilon Q} \left\| \frac{\partial x(A)}{\partial_{k,l}} \right\|_\infty ,$$

where Q is the set of missing comparisons and $\|\cdot\|_\infty$ is the Tchebyshev norm. The most critical comparison index (i,j) is determined by the maximum norm of the vector of $\partial x(A)/\partial_{k,l}$ which corresponds to all missing comparisons.

The previous approach is intuitively plausible but computationally non trivial. Moreover, its effectiveness had not been addressed until recently. In [1] Harker's derivatives approach was tested versus a method which randomly selects the next comparison to elicit. The test problems were generated similarly to the ones described at the end of the previous section. The two methods were also tested in a similar manner as before. To our surprise, the two methods performed in a similar manner. Therefore, the obvious conclusion is that one does not have to implement the more complex derivatives method. It is sufficient to select the next comparison just randomly. Of course, the more comparisons are selected, the better is for the accuracy of the final results. Since the order of comparisons seems not to have an impact, the best strategy is to select as the next comparison the one which is easier for the decision maker to elicit.

Conclusions. Deriving the data for MCDM problems is an approach which requires trade-offs. Thus, it should not come as a surprise that optimization can be used at various stages of this crucial phase in solving many MCDM problems. The previous analysis of some key problems signifies that optimization becomes more critical as the size of the decision problem increases.

Finally, it should be stated here that an in depth analysis of many key issues in multicriteria decision making theory and practice is provided in [18].

See also: **Multi-objective optimization: Pareto optimal solutions, properties; Multi-objective optimization: Interactive methods for preference value functions; Multi-objective optimization: Lagrange duality; Multi-objective optimization: Interaction of design and control; Outranking methods; Preference disaggregation; Fuzzy multi-objective linear programming; Multi-objective optimization and decision support systems; Preference disaggregation approach: Basic features, examples from financial decision making; Preference modeling; Multiple objective programming support; Multi-objective integer linear programming; Multi-objective combinatorial optimization; Bi-objective assignment problem; Multicriteria sorting methods; Financial applications of multicriteria analysis; Portfolio selection and multicriteria analysis; Decision support systems with multiple criteria.**

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EVACUATION NETWORKS

Planning and design of evacuation networks is both a complex and critically important optimization problem for a number of emergency situations. One particularly critical class of examples concerns the *emergency* evacuation of *chemical plants*, *high-rise buildings*, and *naval vessels* due to fire, *explosion* or other emergencies. The problem is compounded because the solution must take into account the fact that human occupants may *panic* during the evacuation, therefore, there must be a well-defined set of *evacuation routes* in order to minimize the sense of panic and at the same time create safe, effective routes for evacuation. The problem is a highly *transient*, *stochastic*, *nonlinear*, *combinatorial optimization programming problem*. We focus on evacuation networks where congestion is a significant problem.

Introduction. *Evacuation* is one of the most perilous, pernicious, and persistent problems faced by humanity. *Hurricanes*, *fires*, *earthquakes*, explosions and other natural and man-made disasters happen on almost a daily basis throughout the world. How can we safely evacuate a collection of occupants within an affected region or facility is the fundamental problem faced in evacuation.

Purpose. The purpose of this article is to both introduce to the reader the problem of evacuation and its manifest nature, and also suggest some alternative approaches to optimize this process. That life-threatening evacuations happen as often as they do is somewhat surprising. That people often do not know how to safely evacuate in time of need is a sad reality. That people must help people plan for evacuation is one of the most important activities of a research scientist.

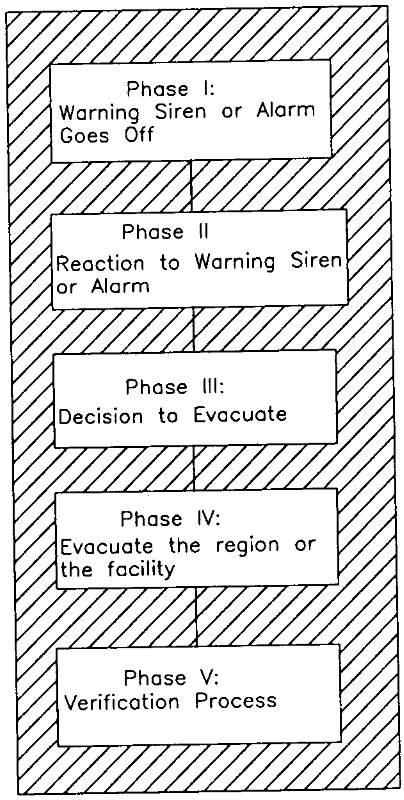

Fig. 1: Processes for an evacuation.

Outline. In this article we first introduce the problem in Section 1 and then describe our fundamental modeling 3-step methodology in Section 2.

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