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Two New Cases of Rank Reversals when the AHP and Some of its Additive Variants are Used that do not Occur with the Multiplicative AHP

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ABSTRACT

Many researchers have long observed some cases in which certain ranking irregularities can occur when the original analytic hierarchy process (AHP), or some of its variants, are used. This paper presents two new categories of ranking irregularities which defy common intuition. These ranking irregularities occur when one decomposes a decision problem into a set of smaller problems each defined on two alternatives and the same criteria as the original problem. These irregularities are possible when the original AHP, or some of its additive variants, are used. Computational experiments on random test problems and an examination of some real-life case studies suggest that these ranking irregularities are dramatically likely to occur. This paper also proves that these ranking irregularities are not possible when a multiplicative variant of the AHP is used. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: multi-criteria decision making; analytic hierarchy process (AHP); ranking reversals; ideal mode AHP; weighted product model; multiplicative AHP

1. BACKGROUND INFORMATION

The analytic hierarchy process (AHP) is a multi-criteria decision making (MCDM) method and was developed by Saaty (1980, 1994). The importance of the AHP and its variants is best illustrated in the more than 1000 references cited in Saaty (1994), in Special Issues (e.g. *Socio-Economic Planning Sciences* **10**(6), 1986; *Mathematical Modelling* **9**(3–5), 1987; *European Journal of Operational Research* **48**(1), 1990; and *Mathematical and Computer Modelling* **17**(4/5), 1993), the ISAHP symposia, and the development of the *Expert Choice* software (www.expertchoice.com).

However, from the early days it was observed that certain ranking irregularities may occur. First, Belton and Gear (1983) noticed that when copies (or near copies) of existing alternatives are introduced in a decision-making problem, then it is possible for the AHP to change the ranking of the alternatives. They attributed this phenomenon to the fact that in the AHP the values of relative performance of the alternatives in terms of each decision criterion in the decision matrix are normalized so they add up to 1.00. They proposed that these values be normalized by dividing them by the largest entry of each column of the decision matrix. When this modification is followed, then the previous type of ranking reversal does not occur. In this paper this variant of the AHP will be called the **revised AHP**. Later, Saaty accepted the previous variant and now it is also called the **ideal mode AHP**. According to Belton (1986) and Belton and Gear (1997) a key issue for the AHP ranking reversals is the interpretation of the criteria weights. Besides the revised AHP, other authors also introduced other variants of the original AHP (see, for instance, Lootsma, 1991, 1993). However, the AHP and some of its variants are considered by many as the most reliable MCDM method.

The fact that rank reversals also occur in the AHP when near copies are considered, has also been studied by Dyer and Ravinder (1983) and Dyer and Wendell (1985). Saaty (1983, 1987) provided some axioms and guidelines on how close a near copy can be to an original alternative without causing a rank reversal. He suggested that the decision maker has to eliminate alternatives from

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consideration that score within 10% of another alternative. This recommendation was later sharply criticized by Dyer (1990a,b). Clearly, these problems are still controversial in decision analysis. Some additional discussions on these truly important issues can be found in Triantaphyllou and Mann (1989, 1994a,b), Harker and Vargas (1990), Saaty (1990), Winkler (1990) and Buede and Maxwell (1995).

The structure of the typical MCDM problem considered in this paper consists of a number, say m, of alternatives and a number, say n, of decision criteria. Each alternative can be evaluated in terms of the decision criteria and the relative importance (or weight) of each criterion can be estimated as well. Let a_{ij} ($i = 1, 2, 3, \ldots m$, and $j = 1, 2, 3, \ldots n$) denote the **performance value** of the i-th alternative (i.e. A_i) in terms of the j-th criterion (i.e. C_j). Also, denote as w_j the **weight** of criterion C_j. That is, the a_{ij} and w_j are the known values of this problem and form, what is called in decision analysis, a **decision matrix**.

This paper is organized as follows. The next section describes the first case of ranking contradiction between the rankings derived when alternatives are compared two at a time and also simultaneously. The third section describes the second case of ranking irregularity. Now the rankings are derived when alternatives are compared two at a time, and it is shown that sometimes they do not follow the transitivity property. The fourth and fifth sections present computational results on randomly generated test problems and also in terms of 22 real life case-studies taken from the literature. A multiplicative variant of the AHP is studied in the sixth section. It is also proved that this variant of the AHP does not suffer of the two new cases of ranking irregularities reported here. Finally, the paper ends with some concluding remarks.

2. CASE 1: RANKING IRREGULARITIES WHEN ALL ALTERNATIVES ARE COMPARED TWO AT A TIME AND ALSO SIMULTANEOUSLY

The main ideas in this section are best described in terms of two illustrative examples. The first example deals with the original AHP and it is presented next.

Example 1: the case of the original AHP

Suppose that the following is the decision matrix and weights of importance of a simple problem with three criteria and the three alternatives A_1, A_2, and A_3:

Alternative	Criteria			Combined priorities
	C_1 (2/7	C_2 2/7	C_3 3/7)	
A_1	9/19	2/12	2/7	0.305
A_2	5/19	1/12	4/7	0.344
A_3	5/19	9/12	1/7	0.351

The above data are assumed to have been derived from **perfectly consistent** judgment matrices with pairwise comparisons. This was assumed in order to block out any effects due to inconsistent pairwise comparisons among the alternatives in terms of each one of the decision criteria. When the original AHP is applied to the above decision matrix, then it can be easily seen that the three alternatives have the priority values shown under the '*combined priorities*' vector. Therefore, in the maximization case, the above combined priorities indicate that the three alternatives are ranked as follows: $A_3 > A_2 > A_1$ (where the symbol '$>$' indicates '*is better than*').

In the following paragraphs the three alternatives are compared in a pairwise fashion. At first, it is assumed that the **criteria weights remain the same as before**. Then, a set of smaller problems are formed by considering two alternatives at a time. In general, there are $m(m-1)/2$ such smaller problems (where m is the number of alternatives). These pairwise comparisons of the alternatives should not be confused with the traditional AHP pairwise comparisons of the alternatives. The traditional pairwise comparisons examine pairs of alternatives in terms of a **single** criterion at a time. The pairwise comparisons described here, refer to small problems that are comprised of two alternatives and the original criteria with the same weights of importance.

The first such sub-problem (which considers the pair of alternatives A_2 and A_3) is described by the following decision matrix:

Problem 1

Alternative	Criteria			Combined priorities	Derived ranking
	C_1 (2/7	C_2 2/7	C_3 3/7)		
A_2	5/10	1/10	4/5	0.5143	$A_2 > A_3$
A_3	5/10	9/10	1/5	0.4857	

The columns of the above matrix have been normalized again, in order to be consistent with the basic requirement of the original AHP (which requires the columns to add up to 1.00).

Similarly, the second and third such subproblem (which consider the pair of alternatives (A_1 and A_2) and (A_1 and A_3), respectively) are as follows:

Problem 2

Alternative	Criteria			Combined priorities	Derived ranking
	C_1 (2/7	C_2 2/7	C_3 3/7)		
A_1	9/14	2/3	2/6	0.5170	$A_1 > A_2$
A_2	5/14	1/3	4/6	0.4830	

Problem 3

Alternative	Criteria			Combined priorities	Derived ranking
	C_1 (2/7	C_2 2/7	C_3 3/7)		
A_1	9/14	2/11	2/3	0.5213	$A_1 > A_3$
A_3	5/14	9/11	1/3	0.4787	

The previous pairwise results, when are taken together, indicate that the ranking of the three alternatives must be: $A_1 > A_2 > A_3$. That is, this ranking is different of the one derived when all the alternatives were considered simultaneously (namely: $A_3 > A_2 > A_1$). As a matter of fact, the second ranking is the **reverse** of the first one. Therefore, the questions which naturally are raised at this point are: **'Which one is the best ranking?'** and **'What are the correct combined priorities of the three alternatives?'**

Before we proceed to answering these questions, we present some further elaborations on this illustrative example. Suppose that the combined priorities of the three alternatives are denoted as P_1, P_2, and P_3, respectively. In the light of this notation, the previous three smaller problems imply that:

$$P_2/P_3 \approx 0.5143/0.4857 = 1.058884$$

$$P_1/P_2 \approx 0.5170/0.4830 = 1.070393$$

$$P_1/P_3 \approx 0.5213/0.4787 = 1.088991$$

In the above relations the approximation symbol '$\approx$' is used instead of equality. This is necessary because if these were equalities, then from the obvious relation: $(P_1/P_2) \times (P_2/P_3) = P_1/P_3$, then the following relation would also **had to be true**: $(1.070393) \times (1.058884) = 1.088911$. However, this **is not true** because the left hand side product is equal to 1.133422. That is, the pairwise comparisons are not always consistent among themselves. These pairwise comparisons lead to the introduction of the following pairwise comparison matrix:

Alternatives	A_1	A_2	A_3	Combined priorities
A_1	1	1.070393	1.088991	0.3505205
A_2	1/1.070393	1	1.058884	0.3318642
A_3	1/1.088991	1/1.058884	1	0.3176152

The above matrix with the pairwise comparisons of the alternatives should not be confused with the well known judgment matrices (i.e. the reciprocal matrices formed with the individual pairwise comparisons elicited in terms of one criterion at a time) produced during the data elicitation process of the AHP. However, this matrix possesses exactly the same mathematical properties as the traditional judgment matrices. In this matrix the alternatives are compared two at a time in terms of all the three criteria combined. This matrix was used to derive the combined priorities depicted next to it.

These combined priorities were derived from the previous pairwise comparisons by applying the Saaty approximation of estimating the right principal eigenvector on the previous reciprocal matrix. This approximation is done by multiplying the elements in each row and then by taking the m-th root (i.e. the third root in this example) (Saaty, 1980). The values thus obtained are then normalized and the '*combined priorities*' vector is obtained. For a critical evaluation of the eigenvector approach and some alternative approaches in processing pairwise matrices the reader may want to refer to Triantaphyllou and Mann (1990, 1994b) and Triantaphyllou *et al.* (1990).

At this point it becomes apparent that the ranking of the three alternatives implied by the last combined priorities is: $A_1 > A_2 > A_3$. Obviously, this ranking is identical to the ranking obtained from the combination of the solutions of the earlier three sub-problems. Again, this ranking is different (reverse) than the ranking obtained when all the alternatives are considered in the traditional manner (as depicted at the beginning of this example). □

A natural thought at this point is whether one should always analyse an AHP problem as in the previous example. The answer to this question is negative. The reason is that sometimes the ranking obtained from the pairwise matrix may be contradictory to the ranking implied by the individual pairwise comparisons. This situation is further demonstrated in the next numerical example.

Example 2: the case of ideal mode AHP

Similarly to the previous example, the case of the ideal mode AHP is best described in terms of an example. Suppose that the following is the decision matrix of a simple problem with three criteria and the three alternatives A_1, A_2, and A_3 (now the data have been divided by the largest entry in each column):

Alternative	Criteria			Combined priorities
	C_1 (4/22	C_2 9/22	C_3 9/22)	
A_1	9/9	5/8	2/8	0.5398
A_2	1/9	8/8	5/8	0.6850
A_3	8/9	2/8	8/8	0.6730

As earlier, the above data are also assumed to have been derived from **perfectly consistent** judgment matrices with pairwise comparisons. When the ideal mode AHP is applied to the above decision matrix, then it can be easily seen that the three alternatives have the priority values shown under the '*combined priorities*' vector. Therefore, in the maximization case, the three alternatives are ranked as follows: $A_2 > A_3 > A_1$.

Working as in the first example, this problem can be decomposed into three sub-problems, each one of which involves two alternatives and all the criteria. These problems, and the corresponding results, are:

Problem 1

Alternative	Criteria			Combined priorities	Derived ranking
	C_1 (4/22	C_2 9/22	C_3 9/22)		
A_2	1/8	8/8	5/8	0.687500	$A_3 > A_2$
A_3	8/8	2/8	8/8	0.693182	

Problem 2

Alternative	Criteria			Combined priorities	Derived ranking
	C_1 (4/22	C_2 9/22	C_3 9/22)		
A_1	9/9	5/8	2/5	0.601136	$A_2 > A_1$
A_2	1/9	8/8	5/5	0.838384	

Problem 3

Alternative	Criteria			Combined priorities	Derived ranking
	C_1 (4/22	C_2 9/22	C_3 9/22)		
A_1	9/9	5/5	2/8	0.693181	$A_3 > A_1$
A_3	8/9	2/5	8/8	0.734343	

The previous results, when are taken together, indicate that the ranking of the three alternatives must be: $A_3 > A_2 > A_1$. This ranking is different of the one derived when all the alternatives were considered simultaneously (namely: $A_2 > A_3 > A_1$).

As in the previous illustrative example, observe that the following relations must be true:

$$P_2/P_3 \approx 0.687500/0.693182 = 0.991803$$

$$P_1/P_2 \approx 0.601136/0.838384 = 0.717018$$

$$P_1/P_3 \approx 0.693182/0.734343 = 0.943948$$

These pairwise comparisons lead to the introduction of the following pairwise comparison matrix:

Alternatives	A_1	A_2	A_3	Combined priorities
A_1	1	0.717018	0.943948	0.2912657
A_2	1/0.717018	1	0.991803	0.3696254
A_3	1/0.943948	1/0.991803	1	0.3391087

The combined priority values (which now add up to one) indicate that the ranking of the three alternatives must be: $A_2 > A_3 > A_1$. However, when the pairwise comparisons in the previous matrix are considered one at a time, then the derived ranking is different. Namely, this ranking is: $A_3 > A_2 > A_1$. This ranking is (by definition) identical to the ranking derived from solving the three smaller sub-problems and then combining the partial solutions. That is, in this case the ranking derived from the combined pairwise matrix is different than the ranking derived when the three smaller sub-problems are considered. Clearly, this was not the case in Example 1. □

Dyer (1990a) claimed that any ranking irregularities will be eliminated if the final stage of the AHP is modified as follows. His suggestion was to subtract the smallest value in each vector in the decision matrix and then divide by the largest remaining element. However, as the following numerical example illustrates, ranking irregularities are still possible to occur.

Example 3: the case of an AHP variant (Dyer, 1990a)

Consider the following problem with four criteria and three alternatives. As before, the a_{ij} data are real (non-normalized) numbers from the continuous interval (1, 9).

Alternative	Criteria			
	C_1 (0.43	C_2 0.12	C_3 0.06	C_4 0.39)
A_1	6.78	7.19	3.28	5.10
A_2	4.26	1.44	8.20	5.76
A_3	5.52	3.99	7.26	8.14

When Dyer's suggestion is applied on the previous matrix, the following decision matrix is derived:

Alternative	Criteria			
	C_1 (0.43	C_2 0.12	C_3 0.06	C_4 0.39)
A_1	1.00	1.00	0	0
A_2	0	0	1.00	0.22
A_3	0.50	0.44	0.81	1.00

Then, the final priorities of the three alternatives are: $P_1 = 0.550$, $P_2 = 0.145$, and $P_3 = 0.707$. Therefore, the suggested ranking of these alternatives is: $A_3 > A_1 > A_2$.

Next, suppose that the three alternatives are examined two at a time (as in the previous examples). It can be easily verified that when the pair A_1 and A_2 is considered and Dyer's suggestion is applied on this smaller problem (which now has two alternatives and four decision criteria), the derived ranking is: $A_1 > A_2$. Similarly, the ranking derived when the pair A_2 and A_3 is considered is: $A_3 > A_2$. Also, when the pair A_1 and A_3 is considered, the ranking becomes: $A_1 > A_3$. Thus, the ranking derived when the alternatives are considered two at a time is: $A_1 > A_3 > A_2$. Obviously, this ranking is in contradiction with the one derived when all the alternatives were considered simultaneously (i.e. $A_3 > A_1 > A_2$). Besides the previous categories of ranking irregularities, a different type of ranking irregularity may occur and it is discussed in the next section. □

3. CASE 2: RANKING IRREGULARITIES WHEN THE ALTERNATIVES ARE COMPARED TWO AT A TIME

The previous discussions clearly raise the issue that when the original or ideal mode AHP are used, then sometimes it may not be obvious what is the correct ranking. Should one accept the ranking derived when all the alternatives and decision criteria are used? Or should one accept the ranking derived from combining the solutions of the smaller problems which consider two alternatives at a time (and all the criteria together)? When the matrix with the combined pairwise comparisons is considered, then may also be the case that one may derive a third ranking which is different than the previous two rankings. Next, it will be shown that the above scenario is possible when the original or ideal mode AHP are used.

One may argue here that when the decision maker encounters a problematic situation as in the previous scenarios, then he/she must be more careful in accepting the final results. A re-evaluation of the characteristics of the problem (especially if a new decision criterion is introduced) may result to more robust data which lead to less ambiguous conclusions. However, as the three illustrative examples discussed in the previous section demonstrate, it is possible to reach contradictory results even when all the pairwise comparisons (i.e. the ones which consider pairs of alternatives in terms of a single criterion) are perfectly consistent. Recall that this was actually the case with the data in the decision matrices in these examples. Therefore, the first question to be answered is which is the right ranking of the alternatives.

Deciding on which one is the right ranking can be an open-ended question. One may argue that the correct ranking is the one derived when all the alternatives and decision criteria are considered simultaneously. However, one may also claim that the solutions of the smaller problems are more reliable because these involve simpler (only two alternatives are considered at a time) decision problems. If one accepts the premise that **simpler problems lead to more reliable solutions**, then the next statement to be accepted is that **the ranking obtained by combining the partial rankings of the smaller problems is the most reliable one**.

Besides the last reason, there is a second argument why the ranking obtained from the smaller problems might be more reliable than the ranking obtained when all the alternatives are considered simultaneously in terms of all the criteria. In the past (see, for instance, Belton and Gear, 1983; Triantaphyllou and Mann, 1989; Dyer 1990a,b), both the original and ideal mode AHP have been sharply criticized because their ranking of the alternatives may change when new or copies of existing alternatives are introduced in a decision problem. These observations formed the basis of many controversial disputes in the scientific and practitioners communities regarding the validity of the AHP method.

When the decision maker accepts the ranking derived by combining the solutions of the smaller problems, then irregularities due to the introduction of copies or near copies of existing alternatives cannot occur. This is obviously true because the ranking of the alternatives is based on how they are ranked when they are considered two at the time. When new alternatives are introduced, then the results of comparing the existing alternatives among themselves remain identical as before the introduction of the new alternatives. Therefore, **the relative ranking of the old alternatives will never change as a result of introducing additional alternatives**.

However, it is possible that the previous sequence of smaller problems **may still result in some ranking abnormalities.** More specifically, when the original or the ideal mode AHP are used for comparing two alternatives at the same time, then it is possible to reach a **new type of ranking contradiction**. This is best illustrated in the following example.

Example 4: the case of the ideal mode AHP

Next we consider an illustrative problem with four criteria and three alternatives. The a_{ij} data are real (non-normalized) numbers from the continuous interval [1, 9].

Alternative	Criteria			
	C_1 (0.27	C_2 0.41	C_3 0.05	C_4 0.27)
A_1	1.92	7.59	1.27	6.13
A_2	3.12	4.31	8.57	7.11
A_3	7.70	4.77	7.45	3.29

Suppose that one uses the ideal mode AHP. When alternatives A_1 and A_2 are compared as in the last two examples, then it is derived that: $A_2 > A_1$. Similarly, when alternatives A_1 and A_3 are compared, then it is derived that: $A_1 > A_3$. That is, these relations suggest that the ranking of the three alternatives must be as follows: $A_2 > A_1 > A_3$. However, when the two alternatives A_3 and A_2 are compared, then the derived ranking is: $A_3 > A_2$. That is, **a logical contradiction** (i.e. failure to follow the **transitivity property** in the derived rankings) is reached. A similar phenomenon can be observed when using the original AHP method. □

4. SOME COMPUTATIONAL RESULTS

The previous illustrative examples clearly demonstrate that certain types of ranking irregularities may occur when the AHP, or some of its variants, are used. In order to gain a deeper understanding on how frequently such irregularities may occur on random test problems, a computational study was undertaken. The data were random numbers from the interval [1, 9] (in order to be compatible with the numerical properties of the Saaty scale). In these test problems the number of alternatives was equal to the following ten different values: 3, 5, 7, . . . 21. Similarly, the number of criteria was equal to 3, 5, 7, . . . 21. Thus, a total of 100 ($= 10 \times 10$) different cases were examined with 10000 replications (in order to derive statistically significant results) per each case. Each random problem was solved using the original and ideal mode AHP. The test problems were treated as the previous illustrative examples. Any ranking irregularity was recorded. Figures 1–6 summarize these results.

Figures 1 and 2 depict how often the indication of the **best alternative** was different when all the alternatives were considered simultaneously and when they were considered in pairs (i.e. similar to the analysis in Examples 1 and 2). Figure 1 refers to the use of the original AHP, while Figure 2 to the use of the ideal mode AHP. Different curves correspond to problems with different numbers of alternatives. As it can be seen from these figures, problems with few alternatives had smaller contradiction rates. The number of decision criteria in a problem seemed to play an insignificant role. Also, these figures show similar contradiction rates for the two versions of the AHP.

On the other hand, Figures 3 and 4 depict contradictions in the ranking of **any alternative**. Now the number of alternatives plays a decisive role, while the number of decision criteria is not as important. Moreover, the contradiction rates are significantly more dramatic. For instance, for problems with five alternatives, the contradiction rates are almost 50%. As before, there is no much difference between the results obtained when the original or ideal mode AHP was used. As it was expected, the contradiction rates in Figures 3 and 4 are much higher than those in Figures 1 and 2. This was expected because the cases of contradiction in the first two figures are naturally included in the results of Figures 3 and 4.

Figures 5 and 6 present the contradiction rates when the alternatives are compared two at a time and cases of logical inconsistencies (i.e. failure to satisfy the transitivity property) were found (as in Example 4). The roles of the number of alternatives and number of decision criteria are similar as before. However, now the ideal mode AHP performs significantly worse than the original AHP.

In all these results problems with less alternatives yielded fewer ranking contradictions than problems with more alternatives. This was expected because the number of pairs of alternatives to be considered for a given case is directly related to the number m of alternatives in the problem (i.e. equal to $m(m-1)/2$). Thus, the chances of finding a logical inconsistency increase accordingly. As it can be seen from the illustrative examples, the number of criteria did not play a prime role. This is also evident in the computational results by the almost horizontal curves.

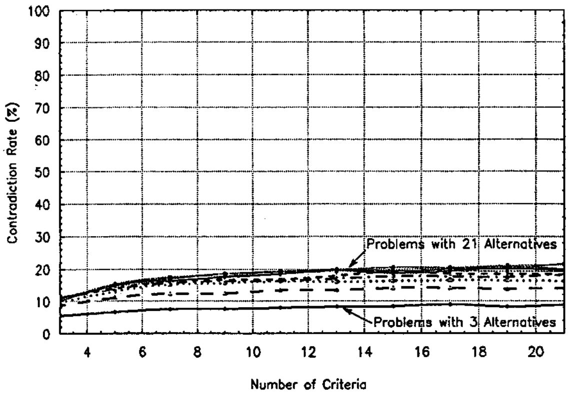

Figure 1. Contradiction rates on the indication of the best alternative when alternatives are considered together and in pairs: the original AHP case.

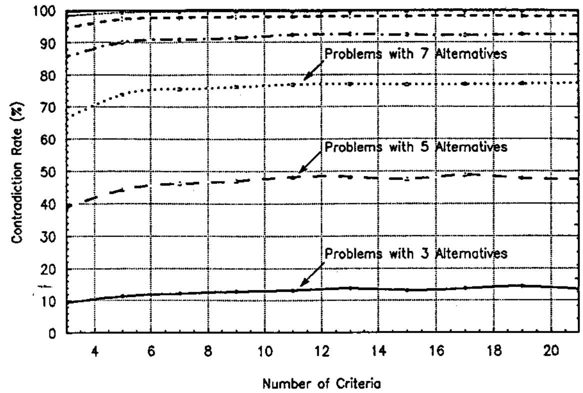

Figure 4. Contradiction rates on the indication of any alternative when alternatives are considered together and in pairs: the ideal mode AHP case.

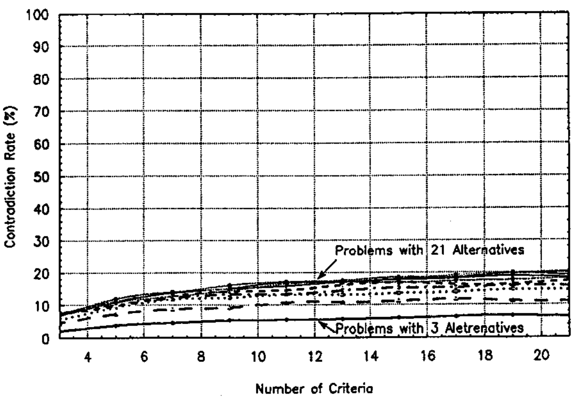

Figure 2. Contradiction rates on the indication of the best alternative when alternatives are considered together and in pairs: the ideal mode AHP case.

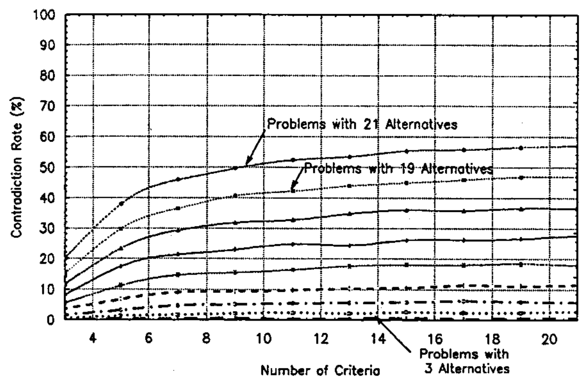

Figure 5. Contradiction rates on the indication of any alternative when alternatives are considered in pairs: the original AHP case.

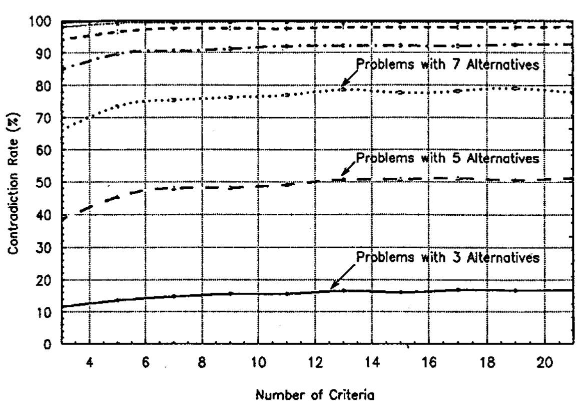

Figure 3. Contradiction rates on the indication of any alternative when alternatives are considered together and in pairs.

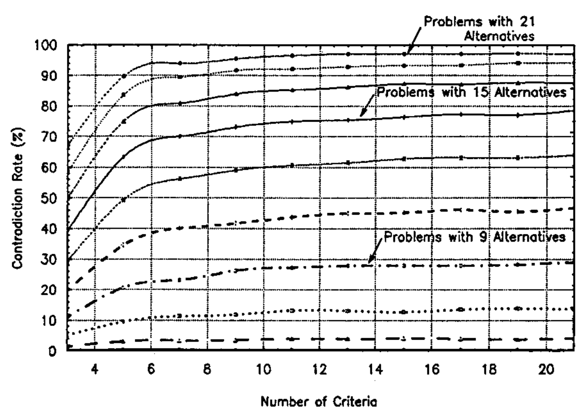

Figure 6. Contradiction rates on the indication of any alternative when alternatives are considered in pairs: the ideal mode AHP case.

Table I. Summary of ranking irregularities in the real-life case studies examined (Part I)

Reference	Domain of application	Size of decision problem		Ranking abnormality present?	
		No. of alternatives	No. of critreria	Original AHP	Ideal mode AHP
Badiru *et al*. (1991)	Manufacturing technology	4	5	Yes (two pairs)	Yes (one pair)
Bard (1986)	Evaluation of space station applications	5	4	Yes (two pairs)	Yes (two pairs)
Bard and Sousk (1989)	Group decision making	3	4	No	Yes (one pair)
Bolster *et al*. (1995) (*first problem*)	Financial analysis	7	17	No	No
Bolster *et al*. (1995) (*second problem*)	Financial analysis	7	17	Yes (one pair)	Yes (one pair)
Boucher and McStravic (1991)	Engineering economy	3	3	Yes (one pair)	Yes (one pair)
Cambron and Evans (1991)	Layout design evaluation	6	10	Yes (one pair)	Yes (one pair)
Hämäläinen (1990) (*first problem*)	Nuclear power management	3	3	No	No
Hämäläinen (1990) (*second problem*)	Nuclear power management	3	3	Yes (one pair)	Yes (one pair)
Hegde and Tadikamalla (1990)	Site selection	7	7	Yes (one pair)	Yes (one pair)
Holguin-Veras (1995)	Highway planning	3	3	Yes (one pair)	Yes (one pair)

Table II. Summary of ranking irregularities in the real-life case studies examined (Part II)

Reference	Domain of application	Size of decision problem		Ranking abnormality present?	
		No. of alternatives	No. of critreria	Original AHP	Ideal mode AHP
Lechner (1995)	Site selection	3	19	No	No
Partovi and Hopton (1994)	Inventory management	3	4	Yes (one pair)	Yes (one pair)
Puelz (1991)	Evaluation of insurance policies	7	4	Yes (two pairs)	Yes (two pairs)
Nydick and Hill (1992)	Purchasing management	4	4	No	No
Saaty (1987)	Policy evaluation	5	4	Yes (one pair)	Yes (one pair)
Saaty (1994)	Bridge evaluation	6	8	Yes (one pair)	Yes (one pair)
Sinuany-Stern (1988)	Sports games	16	6	Yes (one pair)	Yes (four pairs)
Roper-Lowe and Sharp (1990)	Info. technology evaluation	3	15	Yes (one pair)	Yes (two pairs)
Ulengin (1994) (*first problem*)	Traffic management	5	9	No	No
Ulengin (1994) (*second problem*)	Traffic management	5	9	Yes (two pairs)	Yes (two pairs)
Weber (1993)	Automated manufacturing	3	7	Yes (one pair)	Yes (one pair)

5. RESULTS FROM SOME REAL-LIFE CASE STUDIES

The previous empirical analyses revealed that the ranking irregularities studied in this paper may occur frequently in simulated problems. The question which is raised at this point is whether the same could be true with real-life problems. In order to enhance our understanding in this situation, a number of actual case studies were considered. These cases were selected randomly from the published literature. That is, no special screening was performed. The only requirement was to be able to extract the numerical data needed to form a decision matrix. Next, the data were processed as in the previous illustrative examples and any ranking irregularities were recorded.

In all these 22 case studies the two types of ranking irregularities described earlier were studied. The logical inconsistency abnormality (i.e. the one that occurred in the data in example 4) was **not observed** in any one of the 22 real-life case studies examined. Recall that from Figures 1 and 2 this kind of abnormality occurred rather seldomly (i.e. less that 20%) in the random test problems. Therefore, it should not be surprising that it was not detected in these test problems (which were involving few criteria and alternatives). The results are tabulated in Tables I and II.

However, the ranking irregularity which occurs when one compares the ranking derived when all the alternatives are considered and when alternatives are considered two at a time occurs impressively often. Out of the 22 case studies this ranking abnormality occurred 16 times (i.e. 73%) when the original AHP was used and 17 times (i.e. 77%) when the ideal mode AHP was used. Moreover, the number of internal contradictions (i.e. the number of pairs the rankings were different) was often different when one was applying the original or the ideal mode AHP on the same problem (Tables I and II).

6. A MULTIPLICATIVE VERSION OF THE AHP

The use of multiplicative formulas in deriving the relative priorities in decision making is not new (Lootsma, 1991, 1993). A pivotal development is to use multiplicative formulations when one aggregates the performance values a_{ij} with the criteria weights w_j. This is the core step in the weighted product model (which is briefly reviewed in the following sub-section). It is interesting to observe here that Barzilai and Lootsma (1994) have proposed to use a multiplicative variant of the AHP in order to model power relations in group decision making. In this paper we will use a similar approach for single decision-maker problems. Some other developments related to the multiplicative AHP are described in Lootsma (1993, 1999) and Ramanathan (1997).

All the previous problematic situations are caused by the required normalization (either by dividing by the sum of the elements or by the maximum value in a vector) and the use of an **additive function** on the data of the decision matrix. Absolute data cannot be used when the decision criteria are defined on different units (such as dollars, minutes, pounds, etc.). Therefore, one must normalize the pertinent data one way or another. Unfortunately, these normalization steps and the use of additive functions may lead to inconclusive and erroneous results as was the case with the previous numerical examples.

As it was stated in the previous paragraph, if the decision maker deals with qualitative data, or with multiple units of measure, then he/she must use some kind of relative data. That is, the a_{ij} data must be normalized by either dividing by the sum of the entries in that vector, or by the maximum value, or by some other way. However, the use of an additive function can be easily avoided. This is the case, for instance, in the weighted product model (WPM).

Under the WPM approach alternatives are compared two at a time (Miller and Starr, 1969; Chen and Hwang, 1992):

$$R\left(\frac{A_K}{A_L}\right) = \prod_{j=1}^{n} \left(\frac{a_{Kj}}{a_{Lj}}\right)^{W_j} \qquad (1)$$

If the above ratio is greater than or equal to one, then (in the maximization case) the conclusion is that alternative A_K is better than or equal to alternative A_L. Obviously, the best alternative A^* is the one which is better than or at least equal to all other alternatives. Note that the WPM is very similar to the additive function used in the AHP. The WPM is sometimes called **dimensionless analysis** because its structure eliminates any units of measure.

6.1. Ranking consistency under the multiplicative AHP

This sub-section demonstrates that when the WPM procedure (i.e. formula (1)) is used in the last step of the AHP (as it is also recommended in the multiplicative version of the AHP in Barzilai and Lootsma, 1994; p. 162), then the previous ranking irregularities cannot occur. To see this consider any three alternatives, say A_1, A_2 and A_3. Let this problem have n decision criteria. Next, suppose that alternative A_1 is more preferred than alternative A_2. That is, $A_1 > A_2$. Then, according to formula (1) the following relation must be true:

$$R(A_1/A_2) = \prod_{i=1}^{n} \left(\frac{a_{1i}}{a_{2i}}\right)^{W_i} > 1$$

$$\Leftrightarrow \prod_{i=1}^{n} (a_{1i})^{W_i} > \prod_{i=1}^{n} (a_{2i})^{W_i} \quad (2)$$

Similarly with above, now suppose that alternative A_2 is more preferred than alternative A_3. That is, $A_2 > A_3$. Then, according to formula (1) the following relation must be true:

$$R(A_2/A_3) = \prod_{i=1}^{n} \left(\frac{a_{2i}}{a_{3i}}\right)^{W_i} > 1$$

$$\Leftrightarrow \prod_{i=1}^{n} (a_{2i})^{W_i} > \prod_{i=1}^{n} (a_{3i})^{W_i} \quad (3)$$

Relations (2) and (3), when they are combined, yield:

$$\prod_{i=1}^{n} (a_{1i})^{W_i} > \prod_{i=1}^{n} (a_{3i})^{W_i} \qquad \Leftrightarrow \prod_{i=1}^{n} \left(\frac{a_{1i}}{a_{3i}}\right)^{W_i} > 1$$

$$\Leftrightarrow R(A_1/A_3) > 1 \qquad \Leftrightarrow A_1 > A_3 \quad (4)$$

The above analysis demonstrates that if $A_1 > A_2$, and $A_2 > A_3$, then under the proposed multiplicative model, one always gets $A_1 > A_3$. That is, **the transitivity property holds**.

The above proof can easily be generalized to demonstrate that the proposed multiplicative AHP can never yield a ranking abnormality of the form $A_1 > A_2 > \cdots > A_k > \cdots > A_1$. The above considerations, and the assumption that the comparison matrices are perfectly consistent, lead to the proof of the following Theorem 1.

Theorem 1

Under the multiplicative AHP, the derived pairwise rankings always satisfy the transitivity property.

In the original AHP, or in its variants which use additive functions, the previous transitivity property is **not guaranteed**. However, as the previous analysis established, under the multiplicative AHP, this kind of ranking consistency **is always preserved**.

An alternative formulation of the main WPM formula, given as (1), is to consider a single alternative at a time. In this way the relative priorities of the alternatives (and consecutively, their ranking) can be derived. The relative priority, say P_i, of alternative A_i is derived by using the following formula (5) (which is a variant of the previous formula (1)):

$$P_i = \prod_{j=1}^{n} (a_{ij})^{w_j}, \quad \text{for } i = 1, 2, 3, \ldots, m \quad (5)$$

It is noticeable that the **relative** priority values, derived by using the previous formula (5)), are independent of the way the relative performance values a_{ij} have been normalized (assuming that they are different than zero). This is summarized in theorem 2 (which can be easily proved) as follows.

Theorem 2

When the priorities are derived according to formula (5) and then are normalized, they remain independent of the way the relative performance values a_{ij} have been normalized.

The above analyses are further illustrated in terms of a numerical example.

Example 5: an application of the multiplicative AHP

Consider the data used in Example 4 (also presented below).

Alternative	Criteria			
	C_1 (0.27	C_2 0.41	C_3 0.05	C_4 0.27)
A_1	1.92	7.59	1.27	6.13
A_2	3.12	4.31	8.57	7.11
A_3	7.70	4.77	7.45	3.29

When the ideal mode AHP was applied, the ranking derived when two alternatives were considered at a time, had some internal contradictions. Next, the multiplicative formula (1) is applied on these data. The first pair to consider is alternatives A_1 and A_2. The corresponding calculations are:

$$R(A_1/A_2) = \left(\frac{1.92}{3.12}\right)^{0.27} \times \left(\frac{7.59}{4.31}\right)^{0.41} \times \left(\frac{1.27}{8.57}\right)^{0.05} \times \left(\frac{6.13}{7.11}\right)^{0.27} = 0.966 < 1.00$$

Therefore, these alternatives are ranked as follows: $A_2 > A_1$. It can be observed here that the data within the parentheses can be normalized in any possible way, but the values of the ratios will remain unchanged. Next, the pair A_2 and A_3 is considered:

$$R(A_2/A_3) = \left(\frac{3.12}{7.70}\right)^{0.27} \times \left(\frac{4.31}{4.77}\right)^{0.41} \times \left(\frac{8.57}{7.45}\right)^{0.05} \times \left(\frac{7.11}{3.29}\right)^{0.27} = 0.932 < 1.00$$

Therefore, these alternatives are ranked as follows: $A_3 > A_2$. From the last two rankings the following ranking is derived for all three alternatives: $A_3 > A_2 > A_1$. Observe, that when the last pair A_1 and A_3 is considered, then the derived ranking is in agreement with the previous global ranking of the three alternatives. The corresponding calculations for the pair A_1 and A_3 are presented next:

$$R(A_1/A_3) = \left(\frac{1.92}{7.70}\right)^{0.27} \times \left(\frac{7.59}{4.77}\right)^{0.41} \times \left(\frac{1.27}{7.45}\right)^{0.05} \times \left(\frac{6.13}{3.29}\right)^{0.27} = 0.900 < 1.00$$

The above result could also had been obtained by observing that:

$$R(A_1/A_3) = R(A_1/A_2)/A(A_3/A_2) = R(A_1/A_2)/A(A_2/A_3) = 0.966 \times 0.932 = 0.900 < 1$$

This was naturally expected because as it was proved in the previous sub-section, all the ranking irregularities studied in this paper are not possible when the recommended multiplicative version of the AHP is used. Obviously, this was not the case in Example 4, where a ranking irregularity had occurred (it is worth recalling here that the ranking suggested at the beginning of Example 4 was: $A_2 > A_1 > A_3$).

Next, suppose that the relative performance values of the previous three alternatives in terms of the four decision criteria were available. That is, the following decision matrix is assumed to be known:

Alternative	Criteria			
	C_1 (0.27	C_2 0.41	C_3 0.05	C_4 0.27)
A_1	0.151	0.455	0.073	0.371
A_2	0.245	0.259	0.496	0.430
A_3	0.604	0.286	0.431	0.199

When formula (5) is applied on the previous decision matrix, the derived priority values (before normalization) are: $P_1 = 0.735$, $P_2 = 0.742$, and $P_3 = 0.754$. Therefore, the ranking of the three alternatives is: $A_3 > A_2 > A_1$, which is identical (as it should be) with the one found earlier. □

7. CONCLUDING REMARKS

The present study, along with other studies on ranking irregularities, reinforces a growing belief among many decision analysts that ranking irregularities are unavoidable when the AHP is used. This phenomenon seems to be an inherited difficulty when one deals with criteria which are defined on different units. This paper has demonstrated, via some numerical examples, that ranking abnormalities are possible even when the data are perfectly known (and thus perfectly consistent). Therefore, it can very well be the case that the decision maker may never know the exact ranking of the alternatives in a given multi-criteria decision-making problem when the original AHP, or its current additive variants, are used. An extensive computational study and an examination of 22 real-life case studies randomly taken from the open literature demonstrated that these ranking abnormalities may occur dramatically often.

The multiplicative variant of the AHP proposed in (Lootsma, 1993, 1999), and (Barzilai and Lootsma, 1994) is also advocated in this paper.

This is an adaptation of the well known weighted sum model (WPM) (Miller and Starr, 1969; Chen and Hwang, 1992) in the last step of the AHP, in which the decision matrix is processed and the final ranking of the alternatives is determined. In this way, the two new cases of ranking irregularities studied in this paper are not possible. This has been proven theoretically in this paper.

It should be emphasized at this point that the fact that the proposed multiplicative AHP does not possess any of the ranking irregularities studied in this paper does not necessarily guarantee that this method is perfect (i.e. the rankings derived are always the true ones). However, it is obvious that the reverse argument should always be true: That is, a perfect MCDM method should never possess any of the ranking irregularities studied in this paper. Clearly, this is a very important and fascinating issue in decision analysis, and more research is required.

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