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A SENSITIVITY ANALYSIS APPROACH FOR SOME DETERMINISTIC MULTI-CRITERIA DECISION MAKING METHODS

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ABSTRACT

Often data in multi-criteria decision making (MCDM) problems are imprecise and changeable. Therefore, an important step in many applications of MCDM is to perform a sensitivity analysis on the input data. This paper presents a methodology for performing a sensitivity analysis on the weights of the decision criteria and the performance values of the alternatives expressed in terms of the decision criteria. The proposed methodology is demonstrated on three widely used decision methods. These are the weighted sum model (WSM), the weighted product model (WPM), and the analytic hierarchy process (AHP). This paper formalizes a number of important issues on sensitivity analysis and derives some critical theoretical results. Also, a number of illustrative examples and computational experiments further illustrate the application of the proposed methodology.

Subject Areas: ***Analytic Hierarchy Process, Deterministic Decision Making, Multi-Criteria Decision Making, Sensitivity Analysis, Weighted Product Model, and Weighted Sum Model.***

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INTRODUCTION

There is considerable research on sensitivity analysis for some operations research and management science models such as linear programming and investment analysis. For example, in a sensitivity analysis approach for linear programming, Wendel (1992) utilized a tolerance approach to handle variations in the parameters of more than one term (in the LP sense) at a time. Furthermore, that type of sensitivity analysis is considered as a *post-optimality step*. That is, the analysis is done after the optimal decision is determined. However, research on sensitivity analysis in deterministic multi-criteria decision making (MCDM) models is limited.

Myers and Alpert (1968) first introduced the notion of the determinant attributes in choice theory and consumer behavior. In that development Myers and Alpert focused on consumer and behavioral aspects of buyers. Later, Alpert (1971) compared several determinant analysis techniques and found significant support for using the direct dual questioning determinant attribute (DQDA) approach. That made DQDA popular for marketing applications (see, for instance, Anderson, Cox, and Fulcher, 1976; Martin and Winteregg, 1989; and Sinclair and Stalling, 1990).

Barron and Schmidt (1988) recommended two procedures to accomplish a sensitivity analysis in multi-attribute value models. These are an entropy based procedure and a least squares procedure. For the entropy based procedure they assumed nearly equal weights. However, the least squares procedure required a set of arbitrary weights for the attributes. These procedures calculate for a given pair of alternatives, one of which is the best alternative, the *closest* set of weights that equates their ranking. The procedures can also calculate the nearly-equal weights that promote the second best alternative by a specific amount to exceed the *optimal* alternative by a predetermined amount. One of their findings is that in additive models, the weights do matter, that is, for a small change in the weights the optimal alternative may change. Watson and Buede (1987) illustrated sensitivity analysis in a decision modeling strategy.

Von Winterfeldt and Edwards (1986) covered sensitivity analysis in the traditional way for those problems which can be approached by using multi-attribute utility theory (MAUT) or a Bayesian model. They defined the *Flat Maxima Principle* for MAUT problems, which states that the existence of dominance makes sensitivity analysis almost unnecessary. Furthermore, they advice against an overgeneralization of flat maxima, which is applicable exclusively to the expected value and value functions defined on prior probabilities and weights only. One interesting point of the Von Winterfeldt and Edwards approach is the idea of *switchover or break-even points*. A switchover point is the point at which the parameters produce the break-even utility. Switchovers are important because they guide further modeling and elicitation. If the circumstances of the problem imply that both the analysis and the parameters one is using are remote from switchover points, then the decision maker can be confident of the validity of the current results.

Evans (1984) investigated linear programming-like sensitivity analysis in decision theory. His approach is based on the geometric characteristics of optimal decision regions in the probability space. Evans made an analysis on the sensitivity of the optimal decision to changes on probabilities of the states of nature. Also, in Triantaphyllou (1992) a sensitivity analysis approach is described for a class of inventory models. A methodology for sensitivity analysis in multi-objective decision making is described in Ríos Insua (1990). That treatment introduced a general framework for sensitivity analysis which expanded results of the traditional Bayesian approach to decision making. Emphasis is given to cases which use partial and/or doubtful data. Also that work contains an analysis of why the flat maxima principle is not valid. The Ríos Insua (1990) book also includes the description of *SENSATO*; a prototype library of a sensitivity analysis package for decision aids. However, the present paper assumes that the data are not stochastic and it focuses on the issue of sensitivity analysis on the weights of the decision criteria and the performance measures of the alternatives in deterministic environments.

Samson (1988) presented a whole new approach to sensitivity analysis. He proposed that sensitivity analysis should be part of the decision analysis process *thinking in real time*. That is, it should be integrated into every step of the decision analysis. Samson noted that sensitivity analysis can be a most useful tool when it is embedded into a continuous cycle process during which at each stage of the decision process the analysis can go back to previous stages to check, add, or modify parts of the problem.

French (1986), (1989) emphasized the role of sensitivity analysis on decision making. He performed an analysis of the use of interactive decision aids to overcome some of the difficulties in modeling judgments. The models examined were mostly stochastic as opposed to deterministic.

Furthermore, he stressed the importance of having better and more general sensitivity analysis tools. Also, French and Ríos Insua (1989) used a distance minimization approach to determine competitors to a current optimal solution. Some other related sensitivity analysis studies are reported in Alexander (1989); Hartog, Hinloopen and Nijkamp (1989); and Weber, Eisenfuhr and Von Winterfeldt (1988).

A recent development in sensitivity analysis when the analytic hierarchy process (AHP) (see Saaty, 1980, 1994) is used is due to Masuda (1990). In that work Masuda studied the effect of changes on entire vectors of the decision matrix may have on the ranking of the alternatives. That author considered multiple levels of hierarchies. However, he did not offer a procedure for performing a sensitivity analysis on changes on an individual piece of data of a given problem (i.e., on a single criterion weight or the performance value of an alternatives in terms of a given criterion). The proposed sensitivity analysis is complementary to the one developed by Masuda and the two approaches can be used together (since the proposed approach can focus on individual judgments while the Masuda approach considers a single vector at a time). Also, Armacost and Hosseini (1994) presented a procedure for determining the most critical criterion for a single level hierarchy AHP problem. However, the latter work does not explicitly determine what is the smallest change on the current weight of a criterion, such that the existing ranking of the alternatives will change.

As a related comment it should also be stated here that Expert Choice (1990), a software package on the AHP, performs a type of elementary sensitivity analysis. The user has the option to graphically alter the weights of the decision criteria and see on the screen how the rankings of the alternatives will change. However, the issue of criteria sensitivity analysis is not studied systematically. Moreover, Expert Choice does not offer any means for studying the effects of changes on the measures of performance of the alternatives (which is part of the proposed methodology in this paper).

In decision making the weights assigned to the decision criteria attempt to represent the genuine importance of the criteria. When criteria cannot be expressed in quantitative terms (such as cost, weight, volume, etc.), then it is difficult to represent accurately the importance of these criteria. In a situation like this, the decision making process can be improved considerably by identifying the critical criteria (the formal definition is given later) and then re-evaluate more accurately the weights of these criteria. The intuitive belief is that the criterion with the highest weight is the most critical one (Winston, 1991, p. 754). This may not always be true and in some instances the criterion with the lowest weight may be the most critical one.

The decision maker can make better decisions if he/she can determine how critical each criterion is. In other words, how **sensitive** the actual ranking of the alternatives is to changes on the current weights of the decision criteria. In this paper we examine **two closely related sensitivity analysis problems**. In the first problem we determine how critical each criterion is, by performing a sensitivity analysis on the weights of the criteria. This sensitivity analysis approach determines what is the **smallest** change in the current weights of the criteria, which can alter the existing ranking of the alternatives. In the second problem, we use a similar concept to determine how critical the various performance measures of the alternatives (in terms of a single decision criterion at a time) are in the ranking of the alternatives. These two types of sensitivity analysis problems will be explored in latter sections.

The next section briefly describes the MCDM methods considered in this paper. The third section presents the formal definitions of the two sensitivity problems analyzed in this study. The related concepts and methods are further illustrated in terms of some demonstration examples. Computational experiments were also performed in order to increase the insight of these sensitivity issues. Finally, the last section presents the main conclusions of the proposed methodology.

SOME MULTI-CRITERIA DECISION MAKING METHODS

There are three main steps in utilizing a decision making technique involving numerical analysis of a set of discrete alternatives:

1. Determining the relevant criteria and alternatives.

2. Attaching numerical measures to the relative importance (i.e., weights) of the criteria and to the impacts (i.e, the measures of performance) of the alternatives in terms of these criteria.

3. Processing the numerical values to determine a ranking of each alternative.

In this paper we are interested in a sensitivity analysis on the data described in step 2, above.

Consider a decision making problem with M alternatives and N criteria. In this paper alternatives will be denoted as A_i (for $i = 1,2,3,\ldots,M$) and criteria as C_j (for $j = 1,2,3,\ldots,N$). We assume that for each criterion C_j the decision maker has determined its importance, or weight, W_j. It is also assumed that the following relationship is always true:

$$\sum_{j=1}^{N} W_j = 1. \tag{1}$$

Furthermore, it is also assumed that the decision maker has determined a_{ij} (for $i = 1,2,3,\ldots,M$ and $j = 1,2,3,\ldots,N$); the importance (or measure of performance) of alternative A_i in terms of criterion C_j. Then, the core of the typical MCDM problem examined in this paper can be represented by the following **decision matrix** as seen in Table 1.

Table 1: Decision matrix.

	Criterion				
	C_1	C_2	C_3	…	C_N
Alt.	W_1	W_2	W_3	…	W_N
A_1	a_{11}	a_{12}	a_{13}	…	a_{1N}
A_2	a_{21}	a_{22}	a_{23}	…	a_{2N}
A_3	a_{31}	a_{32}	a_{33}	…	a_{3N}
⋮	⋮	⋮	⋮	⋮	⋮
A_M	a_{M1}	a_{M2}	a_{M3}	…	a_{MN}

Some decision methods (for instance, the AHP) require that the a_{ij} values represent **relative** importance. Given the above data and a decision making method, the objective of the decision maker is to find the best (i.e., the most preferred) alternative or to rank the entire set of alternatives.

Let P_i (for $i = 1,2,3,\ldots,M$) represent the **final preference** of alternative A_i when all decision criteria are considered. Different decision methods apply different procedures in calculating the values P_i. Without loss of generality, it can be assumed (by a simple rearrangement of the indexes) that the M alternatives are arranged in such a way that the following relation (ranking) is satisfied (that is, the first alternative is always the best alternative and so on):

$$P_1 \geq P_2 \geq P_3 \ldots \geq P_M. \tag{2}$$

The next subsections briefly describe the three MCDM methods and one variant to be considered in this paper. For an evaluation of these methods the interested reader may want to consult with the analyses reported in Triantaphyllou (1989, 1994).

The Weighted Sum Model

Probably the simplest and still the widest used MCDM method is the weighted sum model (WSM). The preference P_i of alternative A_i (i = 1,2,3,…,M) is calculated according to the following formula (Fishburn, 1967):

$$P_i = \sum_{j=1}^{N} a_{i,j} \, w_j, \qquad \text{for } i = 1,2,3,...,M. \tag{3}$$

Therefore, in the maximization case, the best alternative is the one which corresponds to the largest preference value. The supposition which governs this model is the **additive utility** assumption. However, the WSM should be used only when the decision criteria can be expressed in identical units of measure (e.g., only dollars, or only pounds, or only seconds, etc.).

The Weighted Product Model

The weighted product model (WPM) is very similar to the WSM. The main difference is that instead of addition in the model there is multiplication. Each alternative is compared with the others by multiplying a number of ratios, one for each criterion. Each ratio is raised to the power equivalent to the relative weight of the corresponding criterion. In general, in order to compare alternatives A_p and A_q (where $M \geq p,q \geq 1$) the following product (Bridgman, 1922; Miller and Starr, 1969; Chen and Hwang, 1992) has to be calculated:

$$R\left(\frac{A_p}{A_q}\right) = \prod_{j=1}^{N} \left(\frac{a_{pj}}{a_{qj}}\right)^{w_j}. \tag{4}$$

If the ratio $R(A_p/A_q)$ is greater than or equal to one, then the conclusion is that alternative A_p is more desirable than alternative A_q (for the maximization case). The best alternative is the one which is better than or at least equal to all other alternatives. The WPM is sometimes called **dimensionless analysis** because its structure eliminates any units of measure. Thus, the WPM can be used in single and multi-dimensional decision making problems.

The Analytic Hierarchy Process

Part of the analytic hierarchy process (AHP) (Saaty, 1980, 1994) deals with the structure of an $M \times N$ matrix. This matrix, say matrix $\boldsymbol{A}$, is constructed using the **relative importance** of the alternatives in terms of each criterion. The vector $(a_{i,1}, a_{i,2}, a_{i,3}, \ldots, a_{i,N})$, for each $i = 1,2,\ldots,M$, in this matrix is the principal eigenvector of an $N \times N$ reciprocal matrix which is determined by pairwise comparisons of the impact of the M alternatives on the i-th criterion. Some evidence is presented in Saaty (1980) which supports the technique for eliciting numerical evaluations of qualitative phenomena from experts and decision makers. For a critical evaluation of the eigenvector approach and the AHP the interested reader can consult with the investigations reported by Triantaphyllou, Pardalos and Mann, 1990a, 1990b; and Triantaphyllou and Mann, 1990.

According to AHP the final preference, P_i, of alternative A_i is also given by formula (3). However, now the a_{ij} value expresses the relative performance value of alternative A_i when it is examined with the rest of the other alternatives in terms of criterion C_j. In the maximization case, the best alternative is the one which corresponds to the highest P_i value. The similarity between the WSM and the AHP is clear. The AHP uses **relative** values instead of absolute measures of performance (which may or may not be readily available). In the original version of the AHP the performance values a_{ij} are normalized so they sum up to one. That is, the following relation is always true in the AHP case:

$$\sum_{i=1}^{M} a_{i,j} = 1, \qquad \text{for any } j = 1,2,3,...,N. \tag{5}$$

Thus, it can be used in single or multi-dimensional decision making problems.

Belton and Gear (1983) proposed a revised version of the AHP model. They demonstrated that an unacceptable rank reversal may occur when the AHP is used. Instead of having the relative values of the alternatives A_1, A_2, A_3, …, A_M sum up to one (e.g. equation (5) to hold), they propose to divide each relative value by the maximum quantity of the relative values in each column of the $M \times N$ matrix $\boldsymbol{A}$. Later, Saaty (1994) accepted the previous notion as a variant of the original AHP and now he calls it the

ideal mode AHP.

DESCRIPTION OF THE TWO MAJOR SENSITIVITY ANALYSIS PROBLEMS

The structure of the typical decision problem considered in this paper consists of a number, say M, of alternatives and a number, say N, of decision criteria. Then, the pertinent data form a decision matrix as described earlier at the beginning of the second section of this paper. Given such a decision matrix, the decision problem considered in this paper is how to determine which is the best alternative or rank the entire set of alternatives.

In a simple MCDM situation, all criteria are expressed in terms of the same unit (e.g., dollars). However, in many real life MCDM problems different criteria may be expressed in different dimensions (units). Examples of such dimensions include dollar figures, weight, time, political impact, environmental impact, etc. It is this issue of multiple dimensions which makes the typical MCDM problem to be a complex one.

Given the above data, the objective of the decision maker is to rank the alternatives. The alternatives are ranked according to their final preferences P_i ($i = 1,2,3,\ldots,M$). Recall that the P_i values are calculated according to formulas (3), (4) and (5).

The **first major problem** which is examined in this paper is ***how to determine the most critical criterion*** in the previous decision making problem. Intuitively, one may think that the **most critical criterion** is the criterion which corresponds to the highest weight W_j. However, this notion of criticality may be misleading. In this paper, the most critical criterion is defined in two alternative ways. In the first way the interest is on whether the indication of the best (top) alternative changes or not. On the second definition the interest is on changes on the ranking of any alternative. These definitions are given formally in the next section.

In the previous notion of criticality, the term *smallest change* can be defined in two different ways. The first way is to define smallest change in **absolute terms**. The second way is to define smallest change in **relative terms**. For instance, suppose that the two criteria C_1 and C_2 have weights $W_1 = 0.30$ and W_2

= 0.50, respectively. Furthermore, suppose that when the first weight becomes W'_1 = 0.35, then the existing ranking of the alternatives changes. Similarly, suppose that when the second weight becomes W'_2 = 0.57, then the existing ranking of the alternatives changes. In absolute terms for both criteria, the first criterion is the most critical criterion. This is true since the change of the weights for C_1 is: $|W_1 - W'_1| = 0.05$, while for C_2 it is: $|W_2 - W'_2| = 0.07$. That is, for the first criterion the critical change is smaller than for the second criterion.

However, when one considers **relative terms**, then the previous picture changes. In relative terms, the change of the weights for C_1 is: $|W_1 - W'_1| \times 100/W_1 = 16.67$, while for C_2 it is: $|W_2 - W'_2| \times 100/W_2 = 14.00$. That is, for the second criterion the relative change is smaller than for the first criterion. Therefore, when the relative changes are considered, then the most critical criterion is C_2.

It can be observed that regarding changes on the ranking of the alternatives one may view them from two different perspectives as follows. First, one might be interested to see when a change in the current data causes **any** two alternatives to reverse their existing ranking. However, it is also possible one to be interested **only when the best (top)** alternative changes.

Therefore, a total of four alternative definitions can be considered. These are coded as Absolute Any (AA), Absolute Top (AT), Percent Any (PA), and Percent Top (PT). This approach, however, might be misleading. After all, a change, say by 0.03, does not mean much unless someone is also given the original value. A change of 0.03 is very different if the original value was 0.08 or 0.80. That is, it is more meaningful to use **relative** changes. Therefore, in this paper the emphasis will be on relative (percent) changes and thus all developments are based on relative changes. However, the proposed methodology and illustrative numerical examples presented later, do present how one can also derive changes in absolute terms. It can be noticed that in order for one to derive changes in relative terms, changes in absolute terms need to be calculated first.

The above notion of critical change is used to determine both the most critical criterion (***problem 1***) and the most critical a_{ij} performance measure (***problem 2***). As an extension of determining the most critical criterion (or a_{ij} performance measure), the notion of critical change is used to determine how critical **each** criterion weight W_i and performance measure a_{ij} is. These two problems are examined in more detail in the subsequent sections.

PROBLEM 1: DETERMINING THE MOST CRITICAL CRITERION

Definitions and Terminology

First we consider the case of changes in the current weights of the decision criteria.

DEFINITION 1: Let $\delta_{k,i,j}$ ($1 \leq i < j \leq M$ and $1 \leq k \leq N$) denote the minimum change in the current weight W_k of criterion C_k such that the ranking of alternatives A_i and A_j will be reversed.

Also, define as:

$$\delta'_{k,i,j} = \delta_{k,i,j} \times 100/W_k, \quad \text{for any } 1 \leq i < j \leq M \text{ and } 1 \leq k \leq N. \tag{6}$$

That is, $\delta'_{k,i,j}$ expresses changes in relative terms. As it will be shown later (in Theorem 1) it is possible for a given pair of alternatives and a decision criterion, the critical change to be infeasible.

The most critical criterion is defined in two possible ways (recall that from relations (2) alternative A_1 is always assumed to be the best alternative). The first of these two definitions (i.e., definition 2) applies when one is interested only in changes in the best alternative, while the second definition (i.e., definition 3) applies when one is interested in changes in the ranking of any alternative. Recall that $|s|$ stands for the absolute value function (e.g., $|-5| = +5$).

DEFINITION 2: The **Percent-Top (or PT) critical criterion** is the criterion which corresponds to the smallest $|\delta'_{k,1,j}|$ ($1 \leq j \leq M$ and $1 \leq k \leq N$) value.

DEFINITION 3: The **Percent-Any (or PA) critical criterion** is the criterion which corresponds to the smallest $|\delta'_{k,i,j}|$ ($1 \leq i < j \leq M$ and $1 \leq k \leq N$) value.

It can be recalled that in this paper we adopt the definitions which correspond to relative changes. The following two definitions express how critical a given decision criterion is.

DEFINITION 4: The **criticality degree of criterion C_k**, denoted as D'_k, is the smallest percent amount by which the current value of W_k must change, such that the existing ranking of the alternatives will change. That is, the following relation is true:

$$D'_k = \min_{1 \le i < j \le M} \{ | \delta'_{k,i,j} | \}, \quad \textit{for all} \quad N \ge k \ge 1.$$

DEFINITION 5: The **sensitivity coefficient of criterion C_k**, denoted as $\textbf{\textit{sens}}(C_k)$, is the reciprocal of its criticality degree. That is, the following relation is true:

$$\textbf{\textit{sens}}(C_k) = \frac{1}{D'_k}, \quad \textit{for any} \quad N \ge k \ge 1.$$

If the criticality degree is infeasible (i.e., impossible to change any alternative rank with any weight change), then the sensitivity coefficient is set equal to zero.

The previous two definitions 4 and 5 are based on changes on the ranking of any alternative. However, one may be interested only in changes on the ranking of the best (top) alternative. For instance, in a problem involving the purchase of a house, the focus is on the best house and the ranking of all alternative houses may be of secondary interest. In cases like the above, one may want to use modifications of the criticality degree and sensitivity coefficient concepts in which changes are only defined on the ranking of the best alternative. Also observe that since D'_k is always less than 1.00, it follows that the value of $sens(C_k)$ is always greater than or equal to 1.00.

Some Theoretical Results in Determining the Most Critical Criteria

Case (i): Using the WSM or the AHP Method

Now it is assumed that a decision maker uses the WSM or the AHP method and he/she wishes to alter the existing ranking of alternatives A_1 and A_2 by modifying only the current weight W_1 of criterion C_1. Currently, the following relation is true (as it was assumed in (2)): $P_1 \ge P_2$. In the Appendix it is shown that the minimum quantity $\delta_{1,1,2}$, needed to reverse the current ranking of alternatives A_1 and A_2, should satisfy the following relation:

$$\delta_{1,1,2} < \frac{(P_2 - P_1)}{(a_{21} - a_{11})}, \qquad if \ (a_{21} > a_{11}), \quad or:$$
$$\delta_{1,1,2} > \frac{(P_2 - P_1)}{(a_{21} - a_{11})}, \qquad if \ (a_{21} < a_{11}). \tag{7a}$$

Furthermore, the following condition should also be satisfied for the new weight $W^*_1 = W_1 - \delta_{1,1,2}$ to be feasible:

$$\begin{aligned} 0 &\leq W_1^*, && which \ implies \\ 0 &\leq W_1 - \delta_{1,1,2}, && which \ implies \\ \delta_{1,1,2} &\leq W_1. \end{aligned} \tag{7b}$$

In these developments it is not required to have $W^*_i \leq 1$ because these weights are re-normalized to add up to one.

From relations (7a) and (7b), above, it can be seen that sometimes the value $\delta_{1,1,2}$ **may not** have a feasible value. In other words, it may be **impossible** to reverse the existing ranking of the alternative A_1 and A_2 by making changes on the current weight of criterion C_1. This situation occurs when the value of the ratio:

$$\frac{(P_2 - P_1)}{(a_{21} - a_{11})}$$

is greater than W_1.

The previous considerations can be generalized easily and thus lead to the proof of the following theorem which covers the general case (recall that currently the following relation is assumed to be true from (2): $P_i \geq P_j$, for all $1 \leq i \leq j \leq M$).

THEOREM 1: When the WSM, AHP, or ideal mode AHP methods are used, the quantity $\delta'_{k,i,j}$ ($1 \leq i < j \leq M$ and $1 \leq k \leq N$), by which the current weight W_k of criterion C_k needs to be modified (after normalization) so that the ranking of the alternatives A_i and A_j will be reversed, is given as follows:

$$\delta'_{k,i,j} < \frac{(P_j - P_i)}{(a_{jk} - a_{ik})} \times \frac{100}{W_k}, \qquad if \ (a_{jk} > a_{ik}) \quad or:$$
$$\delta'_{k,i,j} > \frac{(P_j - P_i)}{(a_{jk} - a_{ik})} \times \frac{100}{W_k}, \qquad if \ (a_{jk} < a_{ik}). \tag{8a}$$

Furthermore, the following condition should also be satisfied for the value of $\delta'_{k,i,j}$ to be feasible:

$$\frac{(P_j - P_i)}{(a_{jk} - a_{ik})} \leq W_k. \tag{8b}$$

From the previous considerations it can be seen that if *alternative* A_i ***dominates*** *alternative* A_j *(i.e.,* $a_{ik} \geq a_{jk}$, *for all* $k = 1, 2, \ldots, N$*) then, it is* ***impossible*** *to make alternative* A_j *more preferred than alternative* A_i *by changing the weights of the criteria*. Also, *a criterion* C_k *is a* ***robust criterion*** *if all* $\delta'_{k,i,j}$ *(for* $1 \leq i < j \leq M$ *and* $1 \leq k \leq N$*) quantities associated with it are infeasible*. In other words, if equation (8b) is violated for all $i,j = 1,2,3,\ldots,M$, for some criterion C_k then, any change on the weight of that criterion does not affect the existing ranking of any of the alternatives and thus this criterion is a robust one and consequently it can be dropped from further consideration.

Therefore, if one is interested in determining the **most critical criterion**, then all possible $\delta'_{k,i,j}$ ($1 \leq i < j \leq M$ and $1 \leq k \leq N$) values need to be calculated. Observe that there are $N \times (M(M\text{-}1))/2$ such possible $\delta'_{k,i,j}$ quantities. This issue is further illustrated in the following numerical example.

A Numerical Example for the WSM Case

Consider a decision making problem with the four alternatives A_1, A_2, A_3, and A_4 and the four decision criteria C_1, C_2, C_3, and C_4. Suppose that the following Table 2 is its corresponding decision matrix when the WSM (or the AHP with one hierarchical level) is used. Note that the data were normalized to add up to one, although this is not required by WSM (however, it is required by the AHP).

Table 2: Decision Matrix for the numerical example on the WSM.

	Criterion			
	C_1	**C_2**	**C_3**	**C_4**
Alt.	0.3277	0.3058	0.2876	0.0790
A_1	0.3088	0.2897	0.3867	0.1922
A_2	0.2163	0.3458	0.1755	0.6288
A_3	0.4509	0.2473	0.1194	0.0575
A_4	0.0240	0.1172	0.3184	0.1215

Suppose that we want to apply the WSM (the case of the AHP is identical since the data are already normalized). Then, by using formula(3)) the final preferences and ranking of the four alternatives are as shown in Table 3.

Table 3: Current final preferences.

Alternative	Preference (P_j)	Ranking
A_1	0.3162	1*
A_2	0.2768	2
A_3	0.2621	3
A_4	0.1449	4

Note: * indicates the most preferred (best) alternative.

Therefore, the relation $P_1 \geq P_2 \geq P_3 \geq P_4$ holds and as result the most preferred alternative is A_1. Observe that according to the weights of the four criteria, criterion C_1 **appears** to be the most important one. The minimum change $\delta_{1,1,3}$ needed to alter the current weight W_1 so that the current ranking of the two alternatives A_1 and A_3 will be reversed, can be found by using relation (8a) of Theorem 1 as follows:

$$\delta_{1,1,3} < \frac{(0.2621 - 0.3162)}{(0.4509 - 0.3088)}, \quad or$$

$$\delta_{1,1,3} < -0.3807.$$

The quantity -0.3807 satisfies (8b), as it is less than W_1 (= 0.3277). Thus, the modified weight W^*_1 of the first criterion (before normalization) for this case is:

$$W^*_1 = [0.3277 - (-0.3807)] = 0.7084.$$

Working as above for all possible combinations of criteria and pairs of alternatives, Table 4 is derived. Table 5 depicts the changes in relative terms (that is, the $\delta'_{k,i,j}$ values as computed by using relation (8a) of Theorem 1). Observe that negative changes in Table 4 indicate increases, while positive changes indicate decreases. Also note that the changes (either percentages or in absolute terms) are before normalization. The boldfaced numbers in either table indicate minimum critical changes (as explained in the next paragraphs).

The **Percent-Top (PT) critical criterion** can be found by looking for the smallest relative value of all rows which are related to alternative A_1 (i.e., the best alternative) in Table 5. The smallest such percentage (i.e., 64.8818%) corresponds to criterion C_3 when the pair of alternatives A_1 and A_2 is considered. For criterion C_3 a reduction of its current weight by 64.8818% will make A_2 the most preferred alternative and A_1 will **not** be the best alternative any more.

Table 4: All possible $\delta_{k,i,j}$ values (absolute change in criteria weights).

Pair of Alternatives	**Criterion**			
	C_1	**C_2**	**C_3**	**C_4**
A_1 - A_2	N/F	-0.7023	0.1866	**-0.0902**
A_1 - A_3	-0.3807	N/F	0.2024	N/F
A_1 - A_4	N/F	N/F	N/F	N/F
A_2 - A_3	-0.0627	0.1492	0.0262	**0.0257**
A_2 - A_4	N/F	N/F	-0.9230	N/F
A_3 - A_4	0.2745	N/F	-0.5890	-1.8313

Note: N/F stands for Non-Feasible. That is, the corresponding δ value does not satisfy relation (8b).

The **Percent-Any (PA) critical criterion** can be found by looking for the smallest relative $\delta'_{k,i,j}$ value in the entire Table 5. Such smallest value is $\delta'_{3,2,3}$ = 9.1099% and it (again) corresponds to criterion C_3. Therefore, the **PA critical criterion** is C_3. Finally, observe that it is a coincidence that both definitions of the most **critical criterion** indicate the same criterion (i.e., criterion C_3) in this numerical example.

At this point it should be stated that if a decision maker wishes to define the most critical criterion in absolute changes, then the previous two definitions of Percent-Top (PT) and Percent-Any (PA) critical criterion correspond to **Absolute-Top (AT)** and **Absolute-Any (AA)** critical criterion, respectively. From Table 4 it can be easily verified that the AT criterion is C_4 and also, by coincidence, the AA criterion is C_4 (the corresponding minimum changes are boldfaced). Later, some computational results indicate how frequently various alternative definitions of the most critical criterion may point out to the same criterion.

When definition 4 is used, then from Table 5 it follows that the criticality degrees of the four criteria are: D'_1 = |-19.1334| = 19.1334, D'_2 = 48.7901, D'_3 = 9.1099, and D'_4 = 32.5317. Therefore, the sensitivity coefficients of the four decision criteria (according to definition 5) are: $sens(C_1)$ = 0.0523, $sens(C_2)$ = 0.0205, $sens(C_3)$ = 0.1098, and $sens(C_4)$ = 0.0307. That is, the most sensitive decision criterion is C_3, followed by C_1, C_4, and C_2.

Table 5: All possible $\delta'_{k,i,j}$ values (percent change in criteria weights).

Pair of Alternatives	Criterion			
	C_1	C_2	C_3	C_4
A_1 - A_2	N/F	-229.7	***64.8818***	-114.1772
A_1 - A_3	-116.1733	N/F	70.3755	N/F
A_1 - A_4	N/F	N/F	N/F	N/F
A_2 - A_3	-19.1334	48.7901	***9.1099***	32.5317
A_2 - A_4	N/F	N/F	-320.9	N/F
A_3 - A_4	83.7656	N/F	-204.8	-2,318.10

Case (ii): Using the WPM Method

Suppose that we are interested in determining the quantity $\delta_{1,1,2}$ when the WPM method is used. Recall that according to relation (4) alternative A_1 is more preferred than alternative A_2 when the following ratio is greater than or equal to one:

$$R\left(\frac{A_1}{A_2}\right) = \prod_{j=1}^{N}\left(\frac{a_{1j}}{a_{2j}}\right)^{W_j}. \tag{9}$$

Furthermore, according to (2), it is currently assumed that $P_1 \geq P_2$. Let P_1' and P_2' denote the new preferences of the two alternatives. Then, when the ranking of these two alternatives is reversed, the relation on the preferences becomes: $P_1' < P_2'$. In the Appendix it is shown that the quantity $\delta_{1,1,2}$ must satisfy the following condition:

$$\delta_{1,1,2} > \frac{\log\left(\prod_{y=1}^{N}\left(\frac{a_{1y}}{a_{2y}}\right)^{W_y}\right)}{\log\left(\frac{a_{11}}{a_{21}}\right)}. \tag{10}$$

The last relationship gives the minimum quantity needed to modify the current weight W_1 of criterion C_1 such that alternative A_2 will become more preferred than alternative A_1 (in the maximization case). Similarly as in the previous subsection, this quantity needs to satisfy condition (7b). The previous considerations can be easily generalized and thus lead to the proof of the following theorem:

THEOREM 2: When the WPM method is used, the critical quantity $\delta'_{k,i,j}$ ($1 \leq i < j \leq M$ and $1 \leq k \leq N$), by which the current weight W_k of criterion C_k needs to be modified (after normalization) so that the ranking of the alternatives A_i and A_j will be reversed, is given as follows:

$$\delta'_{k,i,j} > K, \qquad if\ K \geq 0,\ or:$$
$$\delta'_{k,i,j} < K, \qquad otherwise.$$

where K is defined as:

$$K = \frac{\log\left(\prod_{y=1}^{N}\left(\frac{a_{iy}}{a_{jy}}\right)^{W_y}\right)}{\log\left(\frac{a_{ik}}{a_{jk}}\right)} \times \frac{100}{W_k}. \tag{11a}$$

Furthermore, the following constraint should also be satisfied:

$$\delta'_{k,i,j} \leq 100. \tag{11b}$$

Similarly to Theorem 1, in order to determine the most critical criterion a total of 2 ($N \times M(M - 1)/2$) critical changes (i.e., $\delta'_{k,i,j}$ values) need to be calculated. All previous theoretical considerations for the WPM model are further illustrated in the following numerical example.

A Numerical Example for the WPM Case

Consider a decision making problem with the four alternatives A_1, A_2, A_3, and A_4 and the four decision criteria C_1, C_2, C_3, and C_4. Please note that this numerical example is different than the first one (and the two examples which follow), in order to provide a wider exposure of numerical scenarios. Also, the decision matrices are square (i.e., $M = N$) of mere coincidence. The proposed procedures can be applied to any size of decision matrix without any modification at all. Next, suppose that Table 6 depicts the decision matrix for this numerical example.

Table 6: Decision matrix for the numerical example on the WPM.

	Criterion			
	C_1	**C_2**	**C_3**	**C_4**
Alt.	0.4504	0.1231	0.0848	0.3417
A_1	0.9381	0.3501	0.8811	0.5646
A_2	0.7691	0.4812	0.1679	0.9336
A_3	0.9445	0.1138	0.2219	0.0135
A_4	0.1768	0.0221	0.9462	0.1024

Then, by applying the WPM approach (that is, by using formula (4)) the ranking of the four alternatives is as shown in Table 7. The product expressed by (4) for alternative A_1 is greater than one for all possible combinations which include A_1, thus, the most preferred alternative is A_1. Also, according to the weights of the four criteria, criterion C_1 **appears** to be the most important one, because this is the criterion with the highest weight.

Table 7: Current ranking.

Pair of Alternatives A_i - A_j	**(A_i / A_j) Ratio**	**Ranking**
A_1 - A_2	1.0192	A_1 1*
A_1 - A_3	4.6082	A_2 2
A_1 - A_4	5.3062	A_3 3
A_2 - A_3	4.5216	A_4 4
A_2 - A_4	5.2065	
A_3 - A_4	1.1515	

Note: * indicates the most preferred alternative (in the maximization case).

Consider, the minimum quantity needed to alter the current weight W_4, so that the current ranking of the two alternatives A_1 and A_2 will be reversed. This quantity (expressed as %) can be found by using relation (11a) of Theorem 2 as follows:

$$K = \frac{\log\left(\left(\frac{0.9381}{0.7692}\right)^{0.4504}\left(\frac{0.3501}{0.4812}\right)^{0.1231}\left(\frac{0.8811}{0.1679}\right)^{0.0848}\left(\frac{0.5646}{0.9336}\right)^{0.3417}\right)}{\log\left(\frac{0.5646}{0.9336}\right)} \times \frac{100}{0.3417} = -11.04,$$

therefore, the value of $\delta'_{4,1,2}$ should be smaller than K = - 11.04. Note that this is a feasible value since it can easily be verified that it satisfies the constraint given as (11b). In a similar manner, all possible K values can be determined. These values are depicted in Table 8 (the boldfaced number corresponds to the minimum change).

It is interesting to observe that the **PT**, and **PA critical criteria** happened to point to the same criterion, (i.e., criterion C_4). However, the criterion with the highest weight is criterion C_1. Clearly, this is a **counter-intuitive** conclusion. Also, when definition 4 is used, then from Table 8 it follows that the criticality degrees of the four criteria are: D'_1 = 18.69, D'_2 = |-48.44| = 48.44, D'_3 = 13.50, and D'_4 = |-11.04| = 11.04. Therefore, the sensitivity coefficients of the four decision criteria (according to definition 5) are: $sens(C_1)$ = 0.0535, $sens(C_2)$ = 0.0206, $sens(C_3)$ = 0.0741, and $sens(C_4)$ = 0.0906. That is, the most sensitive decision criterion is C_4, followed by C_3, C_1, and C_2.

Table 8: All possible K values for WPM example.

Pair of Alternatives	**Criterion**			
	C_1	**C_2**	**C_3**	**C_4**
A_1 - A_2	21.21	-48.44	13.50	***-11.04***
A_1 - A_3	N/F	N/F	N/F	N/F
A_1 - A_4	N/F	N/F	N/F	N/F
A_2 - A_3	N/F	N/F	N/F	N/F
A_2 - A_4	N/F	N/F	N/F	N/F
A_3 - A_4	18.69	69.97	-114.72	-20.37

Note: N/F stands for non-feasible, i.e., the corresponding value does not satisfy constraint (13b).

SOME COMPUTATIONAL EXPERIMENTS

A computational study was undertaken to study how often the PT and PA critical criteria were the criteria with the highest or with the lowest weight. For that reason, random decision problems were generated and the PT and PA critical criteria were determined. For the AHP case (only) the data for these problems were generated in a manner similar to the procedure used in Triantaphyllou, Pardalos and Mann (1990a), Triantaphyllou, Lootsma, Pardalos and Mann (1994); and Triantaphyllou and Mann (1990). This manner ensures that the problems are generated completely randomly. For the WPM and WSM cases the data were generated randomly from the uniform distribution in the interval [1, 9].

According to the test problem generation approach described in Triantaphyllou, Lootsma, Pardalos and Mann (1994) the data were generated as follows. First a random weight vector *W* was generated such that the ratio of the largest to the smallest element was less than 9 (in order to comply with the values in the Saaty scale). From these weights the entries of the matrix with the actual pairwise comparisons were determined by using the relationship $a_{ij} = w_i/w_j$. It is assumed that the decision maker **does not know** these values. This matrix is called in Triantaphyllou, Lootsma, Pardalos and Mann (1994) the **Real and Continuous Pairwise (RCP)** matrix. However, it is assumed that the decision maker is capable of estimating the entries of the RCP matrix by forming a matrix in which each entry in the RCP matrix is replaced by a number which is as close as possible to the values allowed in the traditional Saaty scale (i.e., the numbers from the set {9, …, 1, 1/2, …, 1/9}). This is called the **Closest and Discrete Pairwise (CDP)** matrix. Next, the eigenvector of the CDP matrix is estimated and the corresponding vector of the decision matrix is formed.

For instance, if the real (and hence unknown) performance values of three alternatives in terms of a single criterion are: (0.77348, 0.23804, 0.23848), then, the (1,3) element of the corresponding RCP matrix is equal to 3.24342 (= 0.77348/0.23848). Thus, the corresponding CDP element will be equal to 3 (because this value is the closest one from the Saaty scale values: {9, …, 1, 1/2, …, 1/9}). More on this approach and some interesting properties of the CDP matrices can be found in Triantaphyllou, Lootsma, Pardalos and Mann (1994).

Two parameters were considered in these test problems. The first parameter was the number of decision criteria. The second parameter was the number of alternatives. The number of decision criteria was equal to 3, 5, 7, …, 21. Similarly, the number of alternatives was equal to 3, 5, 7, …, 21. In this way we formed 100 different combinations of numbers of criteria and alternatives and 1,000 random test problems were generated for each such combination. This simulation program was written in Fortran, using the IMSL library of subroutines for generating random numbers. These results are depicted in Figures 1 to 12.

For each test problem we examined whether the PA or the PT critical criterion was the criterion with the highest or the criterion with the lowest weight. The results of the computational experiments, when the relative (percent) changes are considered, are depicted in Figures 1 to 4. Figures 5 to 8 illustrate the same concepts but when changes are expressed in **absolute** terms. Finally, Figure 9 depicts some specific results when the WPM is used. In the present study, we solved each problem using the WSM, WPM, AHP and ideal mode AHP method.

The four curves in each figure represent the results from each one of the three different MCDM methods used plus one curve for the ideal mode AHP. The most profound observation is that **all MCDM methods generated almost identical results**. This is indicated by the fact that their curves in Figures 1 to 8 are very close to each other.

Figures 1 to 8 indicate that it makes a significant difference whether critical changes are expressed as percent (i.e., in relative terms) or in absolute terms. When changes are expressed as percentages, then more frequently the criterion with the highest weight is the most critical criterion. This is true both when the concept of the critical criterion is defined in terms of changes on the ranking of the top alternative or in terms of changes on the ranking of any alternative. This is evident when one compares Figure 1 with Figure 2 and Figure 3 with Figure 4. The reverse situation occurs when one defines change in absolute changes. That is, now more frequently the most critical criterion is the criterion with the lowest weight. Figures 5 to 8 depict the corresponding results.

As anticipated, the sensitivity importance of any weight (including highest or lowest) reduces gradually as the number of decision criteria in a problem increases. In a matter of fact, when changes are measured in relative terms (i.e., as a percentage), then the lowest weight is hardly ever sensitive in problems with more than 10 criteria (see also Figures 2 and 4). On the other hand, the number of alternatives has only a minor practical influence. This is indicated in Figures 11 and 12 in which the bottom curve corresponds to problems with 3 alternatives and the top curve to problems with 21 alternatives.

The question which is raised at this point is what kind of changes a decision maker should consider: **The ones defined as percentages or the ones defined in absolute terms?** One may argue here that percentage changes are the most meaningful. After all, a change, say of 0.03, does not mean much unless one also considers what was the initial value (for instance, was the initial value equal to 0.95 or to 0.05?).

Figure 9 depicts how frequently the AT and PT definitions pointed out to the same criterion. Please recall that this situation also occurred in some of the illustrative examples analyzed earlier. Similarly, Figure 10 depicts how frequently the AA and PA definitions pointed out to the same criterion. As expected, the frequency of matching the top rank is always higher than matching all ranks. Moreover, this distinction fades away as the number of criteria in a problem increases. Finally, Figure 11 depicts how frequently all alternative definitions (i.e., AT, PT, AA, and PA) pointed out to the same criterion when the WSM model was used (the other models yielded similar results).

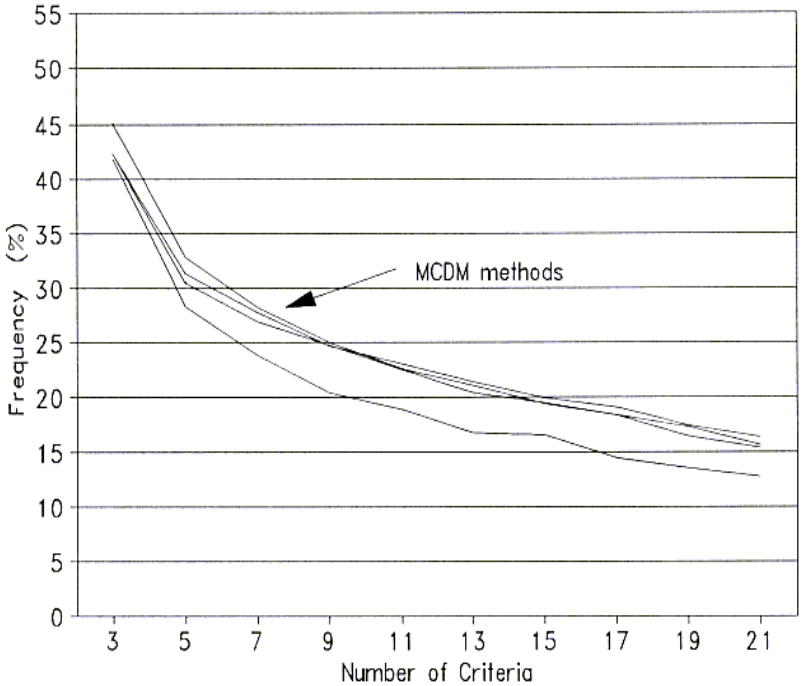

Figure 1. Frequency of the time that the PT critical criterion is the criterion with the ***highest*** weight.

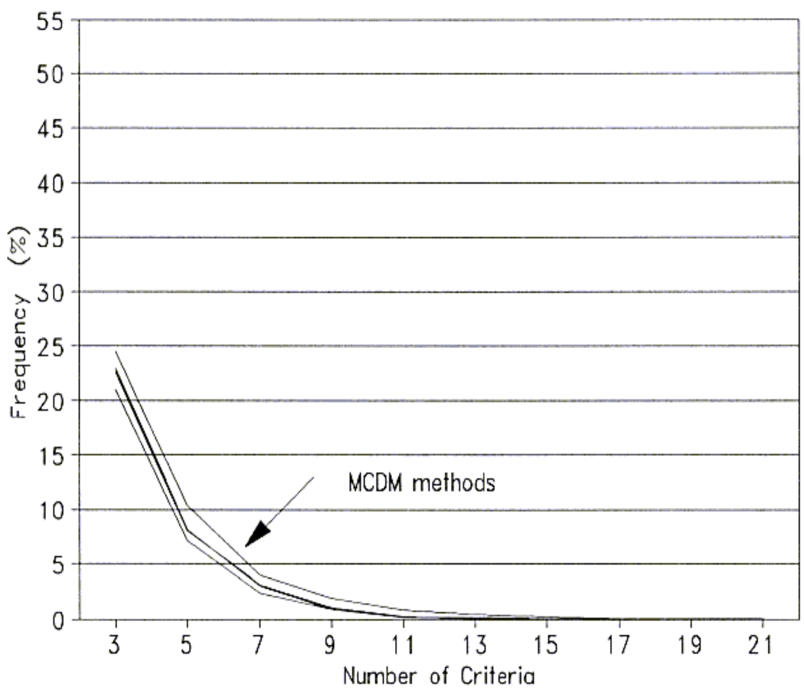

Figure 2. Frequency of the time that the PT critical criterion is the criterion with the ***lowest*** weight.

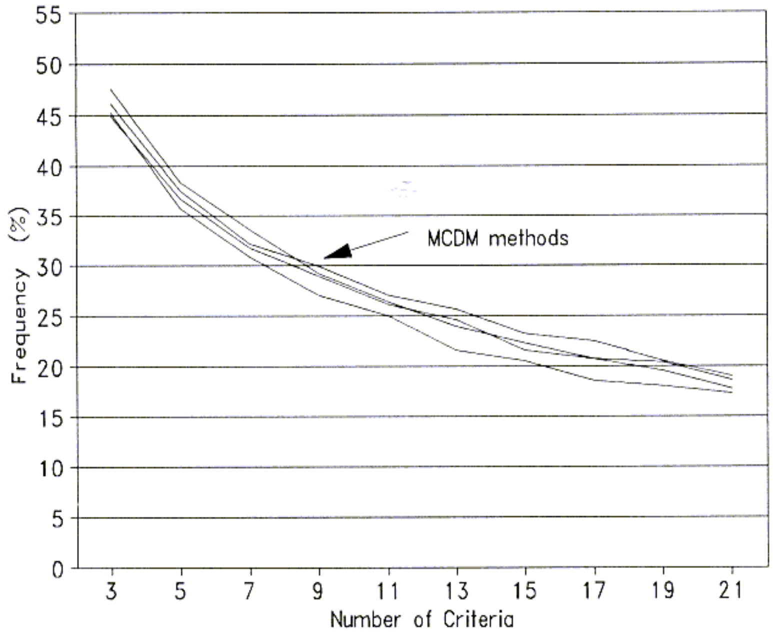

Figure 3. Frequency of the time that the PA critical criterion is the criterion with the ***highest*** weight.

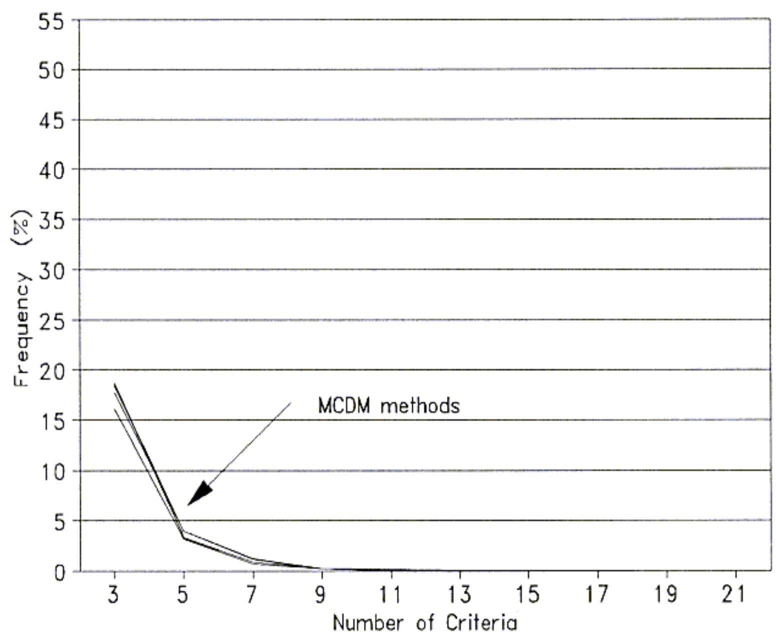

Figure 4. Frequency of the time that the PA critical criterion is the criterion with the ***lowest*** weight.

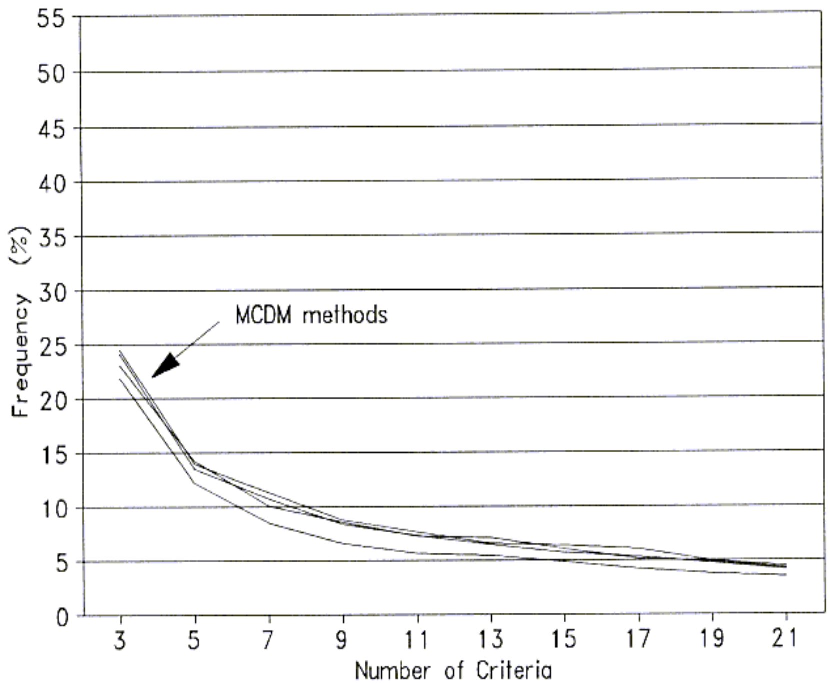

Figure 5. Frequency of the time that the AT critical criterion is the criterion with the ***highest*** weight.

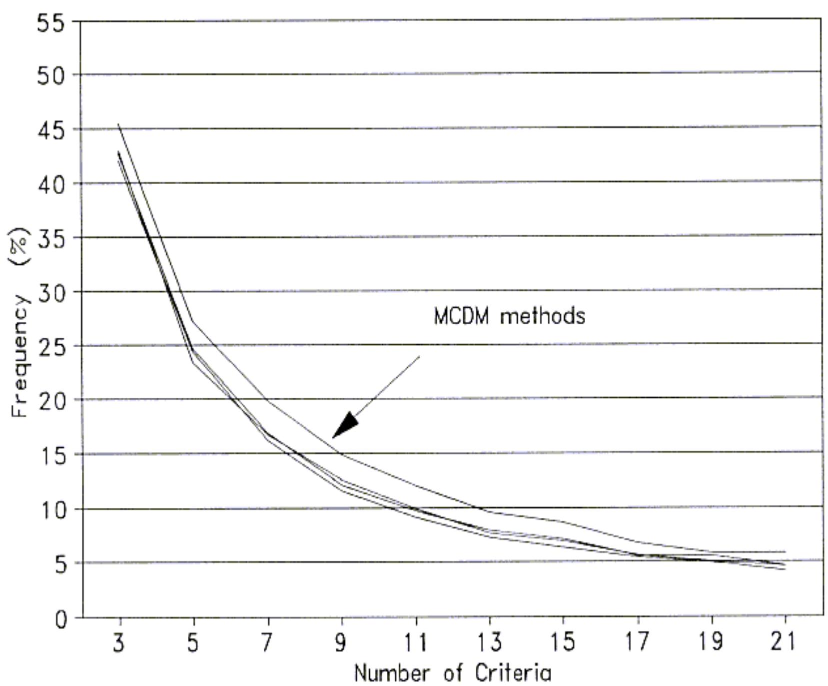

Figure 6. Frequency of the time that the AT critical criterion is the criterion with the ***lowest*** weight.

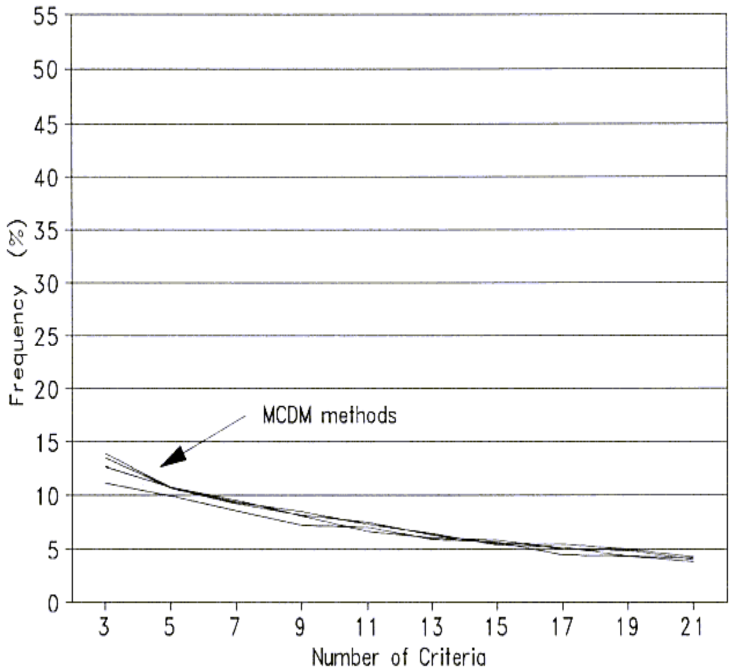

Figure 7. Frequency of the time that the AA critical criterion is the criterion with the ***highest*** weight.

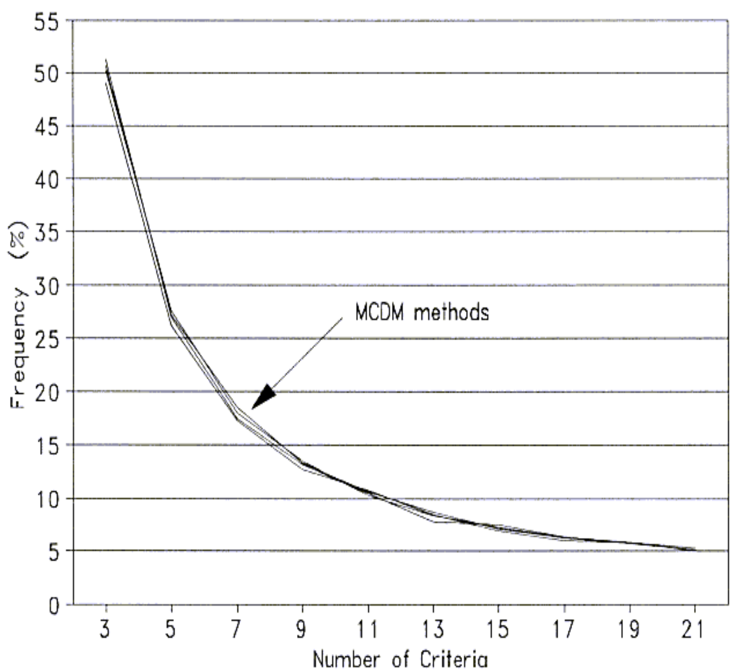

Figure 8. Frequency of the time that the AA critical criterion is the criterion with the ***lowest*** weight.

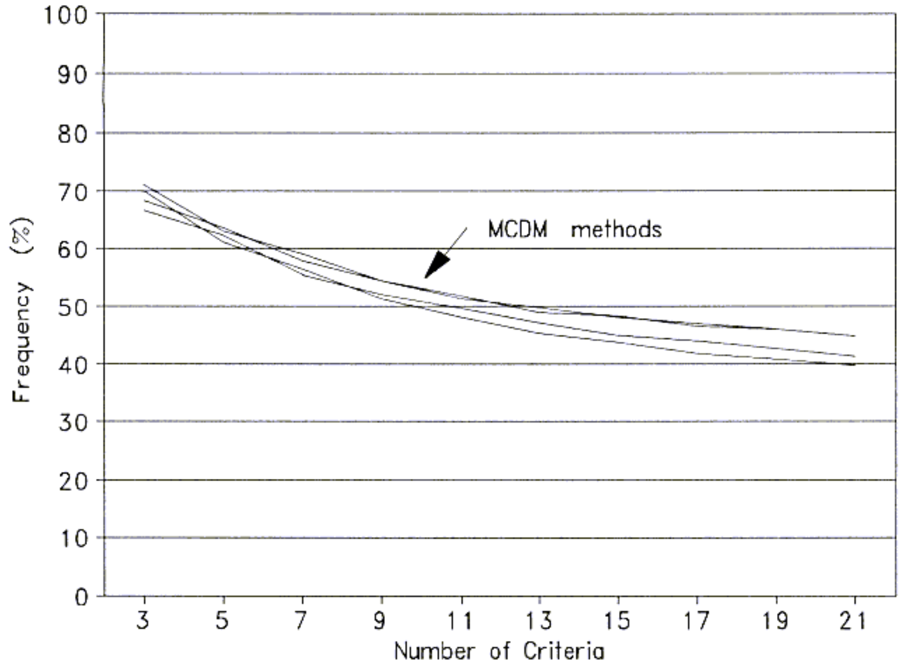

Figure 9. Frequency of the time that the AT and PT definitions point to the ***same*** criterion.

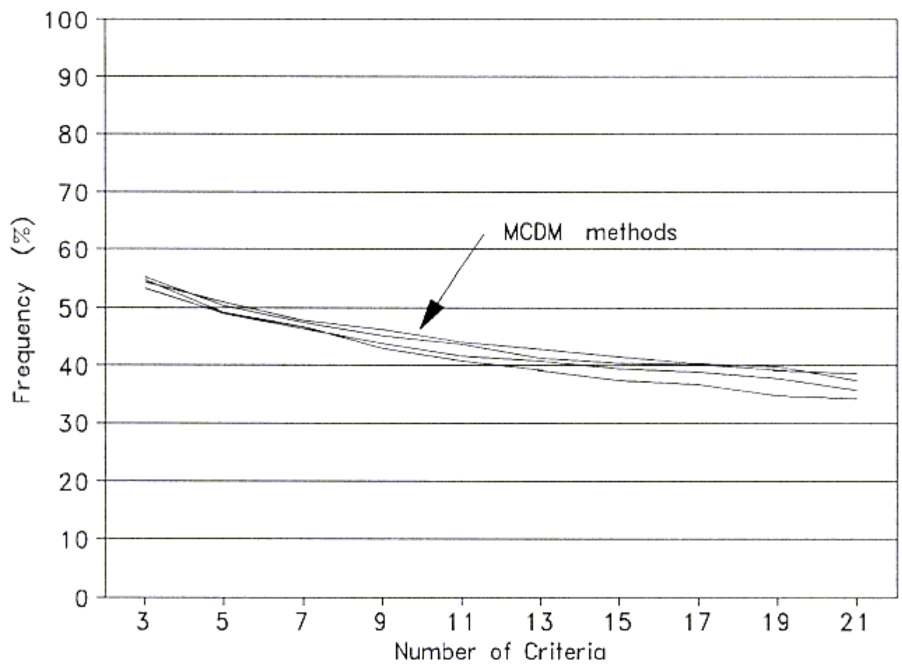

Figure 10. Frequency of the time that the AA and PA definitions point to the ***same*** criterion.

These computational results indicate that the previous coincidence rates are rather high (around 70 to 50 or 40 %) when the number of decision criteria in a problem is rather small (less than 7 or 9). Therefore, if the number of decision criteria is small, one may not have to be concerned on which definition to use. Finally, notice that the number of alternatives in a test problem did not seem to be important. This is indicated by the closeness of the curves (which correspond to problems with different numbers of alternatives) in Figure 12 (which shows a particular set of results when the WPM is used, the other methods yielded similar patterns).

<u>PROBLEM 2:</u>

DETERMINING THE MOST CRITICAL a_{ij} MEASURE OF PERFORMANCE

Definitions and Terminology

The **<u>second major problem</u>** examined in this paper is **<u>how to determine the most critical a_{ij} measure of performance</u>** when the WSM, AHP, or the WPM method is used. The following definitions are pertinent to this problem.

DEFINITION 6: Let ${}_{i,j,k}$ ($1 \leq i < k \leq M$ and $1 \leq j \leq N$) denote the **<u>threshold value of a_{ij}</u>**, which is the minimum change which has to occur on the current value of a_{ij} such that the current ranking between alternatives A_i and A_k will change.

Since there are M alternatives, each a_{ij} performance measure is associated with a total of (M-1) such threshold values. In a similar way as earlier regarding the definition of the $\delta'_{k,i,j}$ values, one can also consider threshold values expressed in relative terms. We denote these relative term threshold values as ${}'_{i,j,k}$. That is:

$$ {}'_{i,j,k} \;=\; {}_{i,j,k} \times 100/a_{ij}, \quad \text{for any } 1 \leq i,\ k \leq M, \text{ and } 1 \leq j \leq N. \qquad (12)$$

For the reasons explained earlier, when we consider threshold values we will mean the ones defined in **<u>relative</u>** terms (i.e., the ${}'_{i,j,k}$ values). Given the previous notion of threshold value, we define as the **<u>most sensitive alternative</u>** the one which is associated with the smallest threshold value. Also as before, one may be interested in changes of the ranking of (only) the best alternative, or in changes in the ranking of any alternative.

As it was mentioned in a previous paragraph, there are (M-1) possible threshold values ${}'_{i,j,k}$ ($i \neq k$, $1 \leq i,\ k \leq M$, and $1 \leq j \leq N$) for any a_{ij} measure of performance (how to calculate these threshold values is given later in theorems 3, 4, and 5). The following three definitions are related to the previous notion of threshold values $^{/}_{i,j,k}$. It can be noticed that analogous definitions are possible if one is interested in changes only on the top alternative (as was the case earlier with definitions 4 and 5).

DEFINITION 7: The **criticality degree of alternative A_i**, denoted as $^{/}_{ij}$, in terms of criterion C_j is the smallest amount (%) by which the current value of a_{ij} must change, such that the existing ranking of alternative A_i will change. That is, the following relation is true:

$$^{/}_{i,j} = \min_{k \neq i} \{ | ^{/}_{i,j,k} | \}, \quad \textit{for all} \;\; M \geq i \geq 1, \;\; \textit{and} \;\; N \geq j \geq 1. \qquad (13)$$

DEFINITION 8: Alternative A_L is the **most critical alternative** if it is associated with the smallest criticality degree. That is, if and only if the following relation is true:

$$^{/}_{Lk} = \min_{M \geq i \geq 1} \left\{ \min_{N \geq j \geq 1} \{ ^{/}_{ij} \} \right\}, \quad \textit{for some} \;\; N \geq k \geq 1. \qquad (14)$$

DEFINITION 9: The **sensitivity coefficient of alternative A_i** in terms of criterion C_j, denoted as ***sens***(a_{ij}), is the reciprocal of its criticality degree. That is, the following condition is true:

$$\textbf{\textit{sens}}(a_{ij}) = \frac{1}{^{/}_{ij}}, \quad \textit{for any} \;\; M \geq i \geq 1, \;\; \textit{and} \;\; N \geq j \geq 1. \qquad (15)$$

If the criticality degree is infeasible, then the sensitivity coefficient is set equal to zero.

From definition 7 it follows that the smaller the criticality degree $^{/}_{ij}$ is, the easier the ranking of alternative A_i can change. Alternatively, definition 9 indicates that ranking changes are easier, as the sensitivity coefficients ***sens***(a_{ij}) are higher. Finally, definition 8, when combined with definitions 7 and 9, indicates that the most sensitive alternative is the one with the **highest** sensitivity coefficient. How to calculate the previous terms is the subject of the next section.

Determining the Threshold Values $^{/}_{\mathbf{i,j,k}}$

Case (i): When Using the WSM or the AHP Method

The Appendix presents the highlights for a proof for Theorem 3. This theorem provides the main formula used to calculate the threshold values $^{/}_{i,j,k}$ and it is stated next.

THEOREM 3: When the WSM method is used, the threshold value $'_{i,j,k}$ (in %) by which the performance measure of alternative A_i in terms of criterion C_j, denoted as $a_{i,j}$, needs to be modified so that the ranking of the alternatives A_i and A_k will be reversed, is given as follows:

$$'_{i,j,k} < R, \qquad when \;\; i < k \;\; or: \qquad\qquad (16a)$$
$$'_{i,j,k} > R, \qquad when \;\; i > k .$$

where R is defined as:

$$R = \frac{(P_i - P_k)}{W_j} \times \frac{100}{a_{ij}} .$$

Furthermore, the following condition should also be satisfied for the threshold value to be feasible:

$$'_{i,j,k} \leq 100 . \qquad (16b)$$

Relation (16b) must hold because from: $0 \leq a_{ij} - {}_{i,j,k}$ the new condition $0 \leq a_{ij} - a_{ij} \times {}_{i,j,k}/100$ is derived which next leads to relation (16b). For the case of the AHP method it can be easily shown (see also the Appendix) that the corresponding theorem is as follows:

THEOREM 4: When the AHP method is used, the threshold value $'_{i,j,k}$ (in %) by which the measure of performance of alternative A_i in terms of criterion C_j needs to be modified so that the ranking of alternatives A_i and A_k will change, is given as follows:

$$'_{i,j,k} = \frac{(P_i - P_k)}{[P_i - P_k + W_j (a_{kj} - a_{ij} + 1)]} \times \frac{100}{a_{ij}} . \qquad (17a)$$

Furthermore, the following condition should also be satisfied for the threshold value to be feasible:

$$'_{i,j,k} \leq 100 . \qquad (17b)$$

The sensitivity analysis of the a_{ij} values, when the WSM model is used, is next demonstrated in terms of a numerical example.

A Numerical Example for the WSM Case

Consider the decision matrix depicted in Table 9 (along with the corresponding final preferences P_i) of an application of the WSM model (that is, the problem has five alternatives and five decision criteria). The AHP case can be developed in an analogous fashion.

Table 9: Decision matrix and initial preferences for numerical example.

	Criterion					
	C_1	**C_2**	**C_3**	**C_4**	**C_5**	
Alt.	0.4146	0.0129	0.2958	0.0604	0.2164	**P_i**
A_1	0.3576	0.2483	0.2899	0.2961	0.3202	0.3244
A_2	0.3603	0.2836	0.0407	0.0939	0.0172	0.1745
A_3	0.0255	0.1745	0.2895	0.2212	0.2641	0.1690
A_4	0.1609	0.2008	0.2960	0.0716	0.0315	0.1680
A_5	0.0957	0.0928	0.0839	0.3172	0.3670	0.1643

Then, when Theorem 4 is used the corresponding $'_{i,j,k}$ threshold values are as in Table 10. The boldfaced entries in Table 10 correspond to the criticality degrees $'_{ij}$ (i.e., the smallest entry per column in each row section, as given in definition 7). The criticality degrees are best summarized in Table 11.

To help interpret the entries in Table 10, consider any one of them, say entry (3,1) (i.e., 89.3). This entry indicates that $'_{1,1,4}$ = 89.3%. That is, the measure of performance a_{11} must be **decreased** by 89.3% from its current value (i.e., 0.3576) to (1 - 0.893) $\times$ 0.3576, in order for alternative A_4 (which is shown on the left right column in this table) to become more preferred than alternative A_1 (note that currently A_1 is more preferred than A_4). A similar interpretation holds for the rest of the entries. Note that some of the entries in the previous table are marked as *N/F*, because they correspond to infeasible values (i.e., condition (17b) in Theorem 4 is violated).

It can be noticed that in Table 10 entries are greater than 100 only when the sign is negative. Recall that negative changes in reality mean increases. If a rating becomes greater than 100, that is all right. In the case of criteria weights the numbers will be re-normalized to add up to 1.00. In the case of the a_{ij} performance measures, the numbers may be re-normalized (for instance, in the AHP model) or may become greater than 1.00 (for instance, in the WPM or WSM models). The boldfaced numbers indicate minimum values.

From Table 11 it follows that the most critical alternatives (according to definition 8) are alternatives A_3 and A_4. This is true because these alternatives correspond to the minimum criticality degrees (equal to 1.1) among all values in Table 11. It can be noticed that the corresponding alternatives from the right column in Table 10 are now within the parentheses in the entries in Table 11. As before, boldfaced numbers represent corresponding minimum values. Finally, Table 12 presents the various sensitivity coefficients (as given in definition 9). Note that if in Table 11 was an infeasible entry (denoted by the "___" symbol), then the corresponding sensitivity coefficient in Table 12 is defined to be equal to 0.

Table 10: Threshold values $^{\prime}_{i,j,k}$ (%) in relative terms for Numerical Example.

Alt. (A_i)	Criterion C_j					Alt. A_k
	C_1	**C_2**	**C_3**	**C_4**	**C_5**	
A_1	**74.1**	N/F	N/F	N/F	N/F	**A_2**
A_1	N/F	N/F	N/F	N/F	N/F	**A_3**
A_1	89.3	N/F	N/F	N/F	N/F	**A_4**
A_1	96.1	N/F	N/F	N/F	N/F	**A_5**
A_2	-157.9	N/F	-1,677.6	N/F	N/F	**A_1**
A_2	5.4	N/F	**35.9**	**79.4**	N/F	**A_3**
A_2	**5.3**	N/F	41.9	N/F	N/F	**A_4**
A_2	8.9	N/F	78.4	N/F	N/F	**A_5**
A_3	-1,535.7	N/F	N/F	N/F	N/F	**A_1**
A_3	-39.3	-355.2	-8.7	-52.5	-13.2	**A_2**
A_3	**8.0**	**38.9**	**1.1**	**8.3**	**2.2**	**A_4**
A_3	41.1	N/F	6.8	29.9	7.3	**A_5**
A_4	-286.1	N/F	N/F	N/F	N/F	**A_1**
A_4	-8.2	N/F	-10.2	-163.0	-98.9	**A_2**
A_4	**-1.7**	**-41.4**	**-1.1**	**-19.7**	**-11.5**	**A_3**
A_4	5.9	N/F	5.3	65.9	40.4	**A_5**
A_5	-460.8	N/F	-970.6	N/F	N/F	**A_1**
A_5	-20.7	N/F	-44.4	-87.4	-21.2	**A_2**
A_5	-12.8	**-544.7**	**-15.9**	-29.7	-6.7	**A_3**
A_5	**10.0**	N/F	18.7	**14.9**	**3.5**	**A_4**

Table 11: Criticality degrees $'_{ij}$ (%) for each a_{ij} performance measure.

Alt.(A_i)	**Criterion C_j**				
	C_1	**C_2**	**C_3**	**C_4**	**C_5**
A_1	74.1(A_2)	___	___	___	___
A_2	5.3(A_4)	___	35.9(A_3)	79.4(A_3)	___
A_3	8.0(A_4)	38.9(A_4)	**1.1(A_4)**	8.3(A_4)	2.2(A_4)
A_4	1.7(A_3)	41.4(A_3)	**-1.1(A_3)**	19.7(A_3)	11.5(A_3)
A_5	10.0(A_4)	544.8(A_3)	15.9(A_3)	14.9(A_4)	3.5(A_4)

Table 12: Sensitivity coefficients ***sens***(a_{ij}) for each a_{ij} performance measure in example.

Alt.(A_i)	**Criterion C_j**				
	C_1	**C_2**	**C_3**	**C_4**	**C_5**
A_1	0.014(A_2)	0	0	0	0
A_2	0.189(A_4)	0	0.028(A_3)	0.013(A_3)	0
A_3	0.125(A_4)	0.026(A_4)	0.909(A_4)	0.121(A_4)	0.455(A_4)
A_4	0.588(A_3)	0.024(A_3)	0.909(A_3)	0.051(A_3)	0.087(A_3)
A_5	0.100(A_4)	0.005(A_3)	0.063(A_3)	0.067(A_4)	0.286(A_4)

Case (ii): When Using the WPM Method

The Appendix also presents the highlights for a proof for Theorem 5. This theorem provides the main formula for calculating the threshold values $'_{i,j,k}$ when the WPM method is used and it is stated next.

THEOREM 5: When the WPM model is used, then the threshold value $'_{i,j,k}$ (in %) by which the performance measure of alternative A_i in terms of criterion C_j, denoted as a_{ij}, needs to be modified so that the ranking of the alternatives A_i and A_k will be reversed, is given as follows:

$$'_{i,j,k} > Q \qquad when\ i > k\ \ or\ : \qquad (18a)$$

$$'_{i,j,k} < Q \qquad when\ i < k\ .$$

and Q is defined as:

$$Q = \left[1 - \sqrt[W_j]{R\left(\frac{A_k}{A_i}\right)} \right] \times 100 .$$

Furthermore, the following condition should also be satisfied for the value to be feasible:

$$'_{i,j,k} \leq 100. \qquad (18b)$$

The application of the previous theorem is also illustrated in the following numerical example.

A Numerical Example When the WPM Method is Used

Consider a decision problem which involves the five alternatives A_1, A_2, A_3, A_4 and A_5 and the five decision criteria C_1, C_2, C_3, C_4 and C_5. Suppose that Table 13 presents its corresponding decision matrix and the WPM model is to be used:

Table 13: Decision matrix for numerical example.

	Criterion				
	C_1	**C_2**	**C_3**	**C_4**	**C_5**
Alt.	0.2363	0.1998	0.0491	0.2695	0.2453
A_1	0.8366	0.5001	0.8179	0.8104	0.6951
A_2	0.4307	0.4782	0.9407	0.2062	0.9259
A_3	0.7755	0.5548	0.6380	0.3407	0.0514
A_4	0.3727	0.7447	0.3214	0.3709	0.0550
A_5	0.4259	0.7126	0.2195	0.0470	0.0014

Recall that in the WPM method normalization of the a_{ij} values is not required. Then, by applying formula (4), the current ranking of the alternatives is as shown in Table 14.

From Table 14 it follows that relation $P_1 \geq P_2 \geq P_3 \geq P_4 \geq P_5$ holds and as result the most preferred alternative is A_1. When Theorem 5 (i.e., formulas (18a) and (18b)) is applied on the previous data, then Table 15 with all possible threshold values $'_{i,j,k}$ is derived. The entries in Table 15 have a similar interpretation as the ones in Table 11.

Table 14: Initial ranking.

Pair of Alternatives	Ratio	Ranking
A_p - A_q	A_p / A_q	
A_1 - A_2	1.580	A_1 1
A_1 - A_3	2.415	A_2 2
A_1 - A_4	2.692	A_3 3
A_1 - A_5	6.152	A_4 4
A_2 - A_3	1.529	A_5 5
A_2 - A_4	1.704	
A_2 - A_5	3.893	
A_3 - A_4	1.115	
A_3 - A_5	2.547	
A_4 - A_5	2.285	

Table 15: Threshold values $'_{i,j,k}$ (%) in relative terms for WPM example.

Alt(A_i)	Criterion Cj					Alt. A_k
	C_1	C_2	C_3	C_4	C_5	
A_1	**85.6**	**89.9**	N/F	**81.7**	**84.5**	A_2
A_1	97.6	98.8	N/F	96.2	97.3	A_3
A_1	98.5	99.3	N/F	97.5	98.2	A_4
A_1	N/F	N/F	N/F	N/F	N/F	A_5
A_2	-593	-887	-1E+06	-446	-546	A_1
A_2	**83.4**	**88.0**	N/F	**79.3**	**82.3**	A_3
A_2	89.5	93.0	N/F	86.2	88.6	A_4
A_2	99.7	99.9	N/F	99.4	99.6	A_5
A_3	-4,072	-8156	-6E+09	-2,538	-3,540	A_1
A_3	-502	-736	-6E+05	-383	-464	A_2
A_3	**36.8**	**41.9**	**89.0**	**33.1**	**35.7**	A_4
A_3	98.1	99.1	N/F	96.9	97.8	A_5
A_4	-6,501	-14,105	-6E+10	-3,844	-5,562	A_1
A_4	-853	-1,339	-5E+06	-622	-777	A_2
A_4	**-58**	**-72**	-811	**-50**	**-56**	A_3
A_4	97.0	98.4	N/F	95.3	96.6	A_5
A_5	-2E+05	-9E+05	-1E+18	-8E+04	-2E+05	A_1
A_5	-3E+04	-9E+04	-1E+14	-2E+04	-3E+04	A_2
A_5	-5,124	-10,672	-2E+10	-3,113	-4,420	A_3
A_5	-3,202	-6,161	-2E+09	-2,049	-2,805	A_4

Some entries in Table 15 are represented in standard exponential format. This happens because they correspond to very high (negative) values. For instance, the entry (8,1), (i.e., -2E + 05) actually represents the value: $-2.0 \times 10^5 = -200{,}000$. It can be argued here that very significant changes (such as the ones represented in exponential format or those which measure in terms of thousands of % change) are not realistic and practically can also be classified as *"N/F"* (e.g., non-feasible) cases. Finally, observe that the highlighted entries in Table 15 correspond to the criticality degrees $/_{ij}$ (as given in definition 7). The criticality degrees are best summarized in Table 16.

Table 16: Criticality degrees $/_{ij}$ (in %) for each a_{ij} measure of performance.

Alt.(A_i)	Criterion C_j				
	C_1	C_2	C_3	C_4	C_5
A_1	85.6(A_2)	89.9(A_2)	___	81.7(A_2)	84.5(A_2)
A_2	83.4(A_3)	88.0(A_3)	___	79.3(A_3)	82.3(A_3)
A_3	36.8(A_4)	41.9(A_4)	89.0(A_4)	**33.1**(A_4)	35.7(A_4)
A_4	58.0(A_3)	72.0(A_3)	___	50.0(A_3)	56.0(A_3)
A_5	___	___	___	___	___

Table 17: Sensitivity coefficients ***sens***(a_{ij}) for each a_{ij} measure of performance in example.

Alt.(A_i)	Criterion C_j				
	C_1	C_2	C_3	C_4	C_5
A_1	0.012(A_2)	0.011(A_2)	0	0.012(A_2)	0.012(A_2)
A_2	0.012(A_3)	0.011(A_3)	0	0.013(A_3)	0.012(A_3)
A_3	0.027(A_4)	0.024(A_4)	0.011(A_4)	0.030(A_4)	0.028(A_4)
A_4	0.017(A_3)	0.014(A_3)	0	0.020(A_3)	0.018(A_3)
A_5	0	0	0	0	0

From Table 16 it follows that the most critical alternative (according to definition 8) is alternative A_3. This is true because this alternative corresponds to the minimum criticality degree (e.g., 33.1) among all values in Table 16. Table 17 presents the various sensitivity coefficients (as defined in definition 9). Note that if in Table 16 was an infeasible entry (denoted by the "___" symbol), then the corresponding sensitivity coefficient in Table 17 is defined to be equal to 0. A comparison of Tables 11 and 16 (or 12 and 17) indicates that the current WPM example is much more **robust** than the AHP example. This is true because the sensitivity coefficients in the WPM example are much smaller. This is a consequence of the specific data used in these two examples and the different nature of the AHP and WPM procedures.

CONCLUDING REMARKS

The contributions of this paper are both theoretical and empirical. This paper presented a unified approach for a sensitivity analysis for three major (and a variant of one) MCDM methods. These methods are: the weighted sum model (WSM), the weighted product model (WPM), and the analytic hierarchy process (AHP) (both in its original and in ideal mode). The proposed sensitivity analysis examines the impact of changes in the weights of importance of the decision criteria (i.e., the W_j values) and the measures of performance of the alternatives in terms of a single decision criterion at a time (i.e., the a_{ij} values) on the final ranking of the alternatives. The theoretical contributions of this paper are best summarized in the five theorems presented in the previous sections.

The empirical contributions are related to the sensitivity analysis of changes in the weights of the decision criteria. We did not cover changes on the a_{ij} values with an empirical study because that would result in too many sensitivity scenarios under consideration for a given problem and thus divert the attention from the central ideas. Recall that for a problem with M alternatives and N criteria there are $M \times N$ different a_{ij} values.

The two most important empirical conclusions of this study are: (*i*) the choice of the MCDM method or number of alternatives has little influence on the sensitivity results; and (*ii*) the most sensitive decision criterion is the one with the highest weight, if weight changes ar measured in relative terms (i.e., as a percentage), and it is the one with the lowest weight if changes are measured in absolute terms.

The main observation of the computational experiments is that the MCDM methods studied here, perform in similar patterns. These patterns refer to the frequency the criterion with the highest (lowest) weight is also the most critical criterion, when changes are measured in percent (absolute) terms.

Moreover, the same results seem to indicate that the number of decision criteria is more important than the number of alternatives in a test problem.

The proposed methodology can be used to carry out a standard sensitivity analysis when one of the previous MCDM methods is used. The benefit of doing a sensitivity analysis is too paramount to be ignored in applications of MCDM techniques to real-life problems. As Dantzig (1963, p. 32) stated it: *"Sensitivity analysis is a fundamental concept in the effective use and implementation of quantitative decision models, whose purpose is to assess the stability of an optimal solution under changes in the parameters."* By knowing which data are more critical, the decision maker can more effectively focus his/her attention to the most critical parts of a given MCDM problem.

Another area of application is during the phase of gathering the data for a MCDM problem, given a limited budget. Often, in real-life applications of MCDM, data are changeable and cannot be precisely determined. In such cases it makes more sense to determine with higher accuracy the weights of the criteria (as well as the a_{ij} measures of performance) which are more critical and with less accuracy the less critical weights. A sensitivity analysis, contacted at an early stage, may reveal which W_j and a_{ij} values have a tendency to be more critical to the final decisions. Therefore, these data can be determined with higher accuracy at a second stage. Next, a new sensitivity analysis cycle can be initiated again. This process can be repeated, in this **stepwise manner**, for a number of times until the entire budget is used or the decision maker is satisfied with the robustness of the results.

The three (and a variant of one) MCDM methods examined in this paper have been fuzzified by Triantaphyllou and Lin (1996). Thus, a natural extension of this research is to develop a sensitivity analysis approach for cases in which the data are fuzzy numbers. An additional area of possible extension is to extend these results to AHP problems with multiple hierarchies.

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APPENDIX

Calculation of the $\delta_{1,1,2}$ Quantity when the AHP or the WSM Method is Used

To help fix ideas suppose that we wish to determine the minimum change on the weight of importance W_1 of criterion C_1 so that the ranking of the two alternatives A_1 and A_2 will be reversed (recall that from (2) currently it is: $P_1 \geq P_2$). As it was introduced in the main part of the paper, $\delta_{k,i,j}$ ($1 \leq i < j \leq M$ and $1 \leq k \leq N$) is the minimum change in the current weight of criterion C_k such that the ranking of alternatives A_i and A_j will be reversed. In the current setting we have: $k = 1$, $i = 1$ and $j = 2$. Therefore, the new (i.e., modified) weight, denoted as W^*_1, of the first criterion is:

$$W_1^* = W_1 - \delta_{1,1,2} . \tag{A1}$$

To preserve property (1) it is necessary that all weights be normalized. Therefore, the new normalized weights, denoted as W'_i, will be as follows:

$$\begin{aligned} W_1' &= \frac{W_1^*}{W_1^* + W_2 + \ldots + W_n} \\ W_2' &= \frac{W_2}{W_1^* + W_2 + \ldots + W_n} \\ &\vdots \\ W_n' &= \frac{W_n}{W_1^* + W_2 + \ldots + W_n} . \end{aligned} \tag{A2}$$

Given the new weights W'_i (for $i = 1, 2, \ldots, N$) it is necessary to express the conditions for the new ranking. Let P'_1 and P'_2 denote the new final preference values for the two alternatives A_1 and A_2, respectively. Since it is desired that the new ranking of the previous two alternatives be reversed, the following relation should be satisfied:

$$P_1' < P_2' . \tag{A3}$$

By making use of the definitions of P'_1 and P'_2 (given as (3) or (5)) when the WSM or the AHP are applied on (A3), the following relation is derived:

$$P_1^{/} = \sum_{j=1}^{N}(W_j^{/}\ a_{1j}) < P_2^{/} = \sum_{j=1}^{N}(W_j^{/}\ a_{2j}), \Rightarrow$$

$$\frac{W_1^{*}\ a_{11}}{W_1^{*} + \sum_{j=2}^{N} W_j} + \frac{\sum_{j=2}^{N} W_j\ a_{1j}}{W_1^{*} + \sum_{j=2}^{N} W_j} < \frac{W_1^{*}\ a_{21}}{W_1^{*} + \sum_{j=2}^{N} W_j} + \frac{\sum_{j=2}^{N} W_j\ a_{2j}}{W_1^{*} + \sum_{j=2}^{N} W_j}, \Rightarrow$$

$$W_1^{*}\ a_{11} + \sum_{j=2}^{N}(W_j\ a_{1j}) < W_1^{*}\ a_{21} + \sum_{j=2}^{N}(W_j\ a_{2j}). \tag{A4}$$

If relations (A4) and (A1) are combined, then the following is derived:

$$-\delta_{1,1,2}\ a_{11} + \sum_{j=1}^{N}(W_j\ a_{1j}) < -\delta_{1,1,2}\ a_{21} + \sum_{j=1}^{N}(W_j\ a_{2j}), \Rightarrow$$

$$-\delta_{1,1,2}\ a_{11} + P_1 < -\delta_{1,1,2}\ a_{21} + P_2, \Rightarrow$$

$$P_1 - P_2 < (a_{11} - a_{21})\ \delta_{1,1,2}, \Rightarrow$$

$$\begin{aligned} \delta_{1,1,2} &< \frac{(P_2 - P_1)}{(a_{21} - a_{11})}, \quad \text{if } (a_{21} > a_{11}) \text{ or :} \\ \delta_{1,1,2} &> \frac{(P_2 - P_1)}{(a_{21} - a_{11})}, \quad \text{if } (a_{21} < a_{11}). \end{aligned} \tag{A5}$$

The above derivations can easily be expanded and are generalized in Theorem 1.

Calculation of the $\delta_{1,1,2}$ Quantity when the WPM Method is Used

As in the previous subsection, suppose that we are interested in determining the quantity $\delta_{1,1,2}$ when the WPM method is used. Recall that from relation (4) alternative A_1 is more preferred than alternative A_2 when the following ratio is greater than or equal to one:

$$R\left(\frac{A_1}{A_2}\right) = \prod_{j=1}^{N}\left(\frac{a_{1j}}{a_{2j}}\right)^{W_j}. \tag{A6}$$

Furthermore, according to (2), it is currently assumed that $P_1 \geq P_2$. Let $P_1^{/}$ and $P_2^{/}$ denote the new preferences of the previous two alternatives. Then, when the ranking of these two alternatives is reversed, the relation on the preferences becomes: $P_1^{/} < P_2^{/}$. Also observe that now the ratio defined in (A6) should be strictly less than one.

By substituting the new weights (i.e., the ones derived after the weight of the first criterion has been modified) into (A6), then the following relations are derived:

$$R'\left(\frac{A_1}{A_2}\right) = \left(\frac{a_{11}}{a_{21}}\right)^{W_1'} \left(\frac{a_{12}}{a_{22}}\right)^{W_2'} \cdots \left(\frac{a_{1N}}{a_{2N}}\right)^{W_N'} < 1, \qquad \Rightarrow$$

$$R'\left(\frac{A_1}{A_2}\right) = r_1^{W_1'} \, r_2^{W_2'} \ . \ . \ . \ r_N^{W_N'} < 1, \qquad where: \tag{A7}$$

$$r_1 = \left(\frac{a_{11}}{a_{12}}\right), \quad r_2 = \left(\frac{a_{12}}{a_{22}}\right), \quad . \ . \ . \ . \ , \quad and \ r_N = \left(\frac{a_{1N}}{a_{2N}}\right).$$

In the previous relations the new normalized weights are as follows:

$$W_1' = \frac{W_1^*}{W^{**}}, \quad W_2' = \frac{W_2}{W^{**}}, \quad . \ . \ . \ . \ , W_N' = \frac{W_N}{W^{**}},$$

$$where: \quad W^{**} = W_1^* + W_2 + \cdots + W_N.$$

Therefore, relation (A7) yields:

$$r_1^{\frac{W_1^*}{W^{**}}} \, r_2^{\frac{W_2}{W^{**}}} \cdots r_N^{\frac{W_N}{W^{**}}} < 1. \tag{A8}$$

Also note that: $W_1^* = W_1 - \delta_{1,1,2}$. Therefore, relation (A8) yields (since the right-hand-side is greater than or equal to zero):

$$\delta_{1,1,2} > \frac{\log\,(r_1^{W_1} \, r_2^{W_2} \cdots r_N^{W_N})}{\log r_1}.$$

When (A7) is used on the previous expression, the following relation is obtained after some simple algebraic manipulations:

$$\delta_{1,1,2} > \frac{\log\left(\prod_{y=1}^{N}\left(\frac{a_{1y}}{a_{2y}}\right)^{W_y}\right)}{\log\left(\frac{a_{11}}{a_{21}}\right)}. \tag{A8}$$

The above derivations can easily be expanded and are generalized in Theorem 2.

Calculation of the $_{3,4,2}$ Quantity when the WSM is Used

The WSM method is an additive multi-attribute model in which the final preference P_i of alternative A_i is obtained according to relation (3). Let P'_i denote the new preferences, that is, the ones after the change on the a_{ij} measure of performance. As before, let us assume that we are interested in reversing the actual ranking between alternatives A_2 and A_3, which is currently from (2): $P_2 \geq P_3$. Furthermore, to get that rank reversal one must find the threshold value $_{3,4,2}$ of the a_{34} measure of performance required to get the following relation:

$$P'_2 < P'_3. \tag{A9}$$

When relation (3) is applied to (A9), one gets have the following:

$$a_{21}W_1 + \ldots + a_{2N}W_N < a_{31}W_1 + \ldots + a'_{34}W_4 + \ldots + a_{3N}W_N, \Rightarrow$$

$$(\text{since } a'_{34} = a_{34} - {}_{3,4,2}) \; P_2 < - {}_{3,4,2} W_4 + P_3.$$

Therefore, in order to reverse the ranking of alternatives A_2 and A_3 by modifying the a_{34} measure of performance, the threshold value $_{3,4,2}$ should satisfy the following relation:

$$_{3,4,2} < \frac{(P_3 - P_2)}{W_4}.$$

Furthermore, the following condition should also be satisfied for the new a'_{34} value to have a feasible meaning:

$$0 \leq a'_{34}, \Rightarrow$$

$$0 \leq a_{34} - {}_{3,4,2}, \Rightarrow$$

$$_{3,4,2} \leq a_{34}.$$

From the previous relations, it can be seen that the $_{3,4,2}$ value must be less than or equal to a_{34} in order to have a feasible value.

Next, assume that the measure of performance to alter is again a_{34}, but now we want to reverse the ranking between alternatives A_3 and A_5 (observe that now: $P_3 \geq P_5$ from (2)). Then, by following an approach similar to the previous one, we get that in order to reverse the current ranking of alternatives A_3 and A_5 the threshold value $_{3,4,5}$ should satisfy the following relation:

$$_{3,4,5} > \frac{(P_3 - P_5)}{W_4}.$$

Moreover, the following condition should be satisfied for the new a'_{34} value to have a feasible meaning:

$$_{3,4,5} \leq a_{34}.$$

The above derivations can easily be expanded and are generalized in Theorem 3.

Calculation of the $'_{i,j,k}$ Quantity when the AHP is Used

Suppose that the interest is in reversing the ranking between alternatives A_i and A_k. As it was stated in definition 6, the threshold value $_{i,j,k}$ ($i \neq k$, $1 \leq i, k \leq M$ and $1 \leq j \leq N$) is the minimum change in the current value of the a_{ij} measure of performance, such that the ranking between alternatives A_i and A_k will be reversed. Let $a^*_{i,j}$ denote the modified a_{ij} measure of performance. That is:

$$a^*_{ij} = a_{ij} - {}_{i,j,k}, \tag{A10}$$

For an easy demonstration of the main ideas, suppose that we are interested in reversing the current ranking between alternatives A_2 and A_3, (where $P_2 \geq P_3$ from (2)) by altering the a_{34} value (only). Therefore, it is necessary to determine the threshold value $_{3,4,2}$ required to get the following relation (where P'_2 and P'_3 denote the new final preference values of alternatives A_2 and A_3, respectively):

$$P'_2 < P'_3, \tag{A11}$$

Recall that the AHP requires normalization of the a_{ij}. Also note that a^*_{34} (= a_{34} - $_{3,4,2}$) denotes the altered value of a_{ij}. Therefore the new a'_{ij} values are as follows:

$$\begin{aligned}
a'_{14} &= \frac{a_{14}}{a_{14} + a_{24} + a^*_{34} + \ldots + a_{M4}} \\
a'_{24} &= \frac{a_{24}}{a_{14} + a_{24} + a^*_{34} + \ldots + a_{M4}} \\
a'_{34} &= \frac{a^*_{34}}{a_{14} + a_{24} + a^*_{34} + \ldots + a_{M4}} \\
&\vdots \\
a'_{M4} &= \frac{a_{M4}}{a_{14} + a_{24} + a^*_{34} + \ldots + a_{M4}} .
\end{aligned} \tag{A12}$$

Furthermore, the denominator of relations (A12) can be simplified by using relation (A10) as follows:

$$a_{14} + a_{24} + a_{34} - {}_{3,4,2} + a_{44} + \ldots + a_{M4} = - {}_{3,4,2} + a_{14} + \ldots + a_{M4} .$$

The above expression can be further reduced to (A13):

$$1 - {}_{3,4,2} \tag{A13}$$

Therefore, relations (A12) can be expressed as follows:

$$a_{14}^{/} = \frac{a_{14}}{1 - \quad _{3,4,2}}$$

$$a_{24}^{/} = \frac{a_{24}}{1 - \quad _{3,4,2}}$$

$$a_{34}^{/} = \frac{a_{34}^{*}}{1 - \quad _{3,4,2}} = \frac{a_{34} - \quad _{3,4,2}}{1 - \quad _{3,4,2}}, \quad (from \ (A.10)) \tag{A14}$$

$$\vdots$$

$$a_{M4}^{/} = \frac{a_{M4}}{1 - \quad _{3,4,2}}.$$

Consequently, by applying relations (A14) and (2) on (A11) we have the following:

$$a_{21}W_1 + \ldots + a_{24}^{/}W_4 + \ldots + a_{2N}W_N < a_{31}W_1 + \ldots + a_{34}^{/}W_4 + \ldots + a_{3N}W_N, \quad \Rightarrow$$

$$a_{21}W_1 + \ldots + a_{24}^{/}W_4 + (a_{24} - a_{24})W_4 + \ldots + a_{2N}W_N <$$

$$a_{31}W_1 + \ldots + a_{34}^{/}W_4 + (a_{34} - a_{34})W_4 + \ldots + a_{3N}W_N, \quad \Rightarrow$$

$$a_{24}^{/}W_4 - a_{24}W_4 + a_{21}W_4 + \ldots + a_{2N}W_N < a_{34}^{/}W_4 - a_{34}W_4 + a_{31}W_4 + \ldots + a_{3N}W_N, \quad \Rightarrow$$

$$a_{24}^{/}W_4 - a_{24}W_4 + P_2 < a_{34}^{/}W_4 - a_{34}W_4 + P_3. \tag{A15}$$

When we substitute relations (A14) on (A15) we get:

$$\frac{a_{24}W_4}{(1 - \quad _{3,4,2})} - a_{24}W_4 + P_2 < \frac{(a_{34} - \quad _{3,4,2})W_4}{(1 - \quad _{3,4,2})} - a_{34}W_4 + P_3,$$

which can be further reduced to (observe that the denominator on the right-hand-side is always a positive number):

$$_{3,4,2} < \frac{P_3 - P_2}{[P_3 - P_2 + W_4(a_{24} - a_{34} + 1)]}. \tag{A16}$$

Furthermore, the following condition should also be satisfied for the new a'_{34} value to have a feasible meaning:

$$0 \leq a_{34}^{/} \leq 1 \quad or \ (from \ (A10)):$$

$$_{3,4,2} \leq a_{34}.$$

From the previous relations, it can be seen that the $_{3,4,2}$ quantity must be within the range $[a_{34} - 1, \ a_{34}]$ in order to have a feasible value.

Now assume that the measure of performance to alter is again a_{34}, but now we want to reverse the ranking between alternatives A_3 and A_5 (note that from (2) now we have: $P_3 \geq P_5$). Then, by following a similar approach as previously one can get that in order to reverse the current ranking of alternatives A_3 and A_5, by modifying the a_{34} measure of performance, the minimum quantity ${}_{3,4,5}$ should satisfy the following relation:

$$ {}_{3,4,5} \; > \; \frac{P_3 - P_5}{[P_3 - P_5 + W_4 (a_{54} - a_{34} + 1)]} \; . $$

The following condition should also be satisfied for the new a'_{34} value to have a feasible meaning:

$$ {}_{3,4,5} \; \leq \; a_{34} \; . $$

Theorem 4 presents a generalization of the previous considerations.

Using the WPM Method for Calculating ${}_{3,4,2}$

In this subsection we follow a similar approach as in the previous subsections. In the WPM model the ratio $R(A_i/A_k)$ should be greater than or equal to 1.00 (according to relation (4)) in order for alternative A_i to be more preferred than alternative A_k. Let's denote as $R'(A_i/A_k)$ the new ratio after the $\delta_{i,j,k}$ change has occurred on the a_{ij} measure of performance. The new ratio $R'(A_i/A_k)$ should be strictly less than 1.00. We want to find the threshold value ${}_{3,4,2}$ in order to get a rank reversal between alternatives A_2 and A_3. In this case the new ratio $R'(A_2/A_3)$ will be as follows:

$$ R' \left(\frac{A_2}{A_3} \right) = \left(\frac{a_{21}}{a_{31}} \right)^{W_1} \times \ldots \times \left(\frac{a_{24}}{a'_{34}} \right)^{W_4} \times \ldots \times \left(\frac{a_{2N}}{a_{3N}} \right)^{W_N} < 1.00 . \qquad \text{(A17)} $$

If we use the relation: $a'_{34} = a_{34} - {}_{3,4,2}$ in (A17), we get:

$$ \left(\frac{a_{21}}{a_{31}} \right)^{W_1} \times \ldots \times \left(\frac{a_{24}}{a_{34} - {}_{3,4,2}} \right)^{W_4} \times \left(\frac{a_{34}}{a_{34}} \right)^{W_4} \times \ldots \times \left(\frac{a_{2N}}{a_{3N}} \right)^{W_N} < 1.00 , $$

or:

$$ \left(\frac{a_{34}}{a_{34} - {}_{3,4,2}} \right)^{W_4} \times R \left(\frac{A_2}{A_3} \right) < 1.00 . $$

After some simple algebraic manipulations we get:

$$ {}_{3,4,2} \; > \; a_{34} \times \left[1 - \sqrt[W_4]{R \left(\frac{A_2}{A_3} \right)} \right] . \qquad \text{(A18)} $$

Furthermore, the following condition should also be satisfied for the new a'_{34} value to be feasible:

$$0 \leq a_{34}, \quad \Rightarrow$$

$$0 \leq a_{34} - {}_{3,4,2}, \quad \Rightarrow$$

$${}_{3,4,2} \leq a_{34}.$$

Now, assume that the measure of performance to alter is again a_{34}, but now we want to reverse the ranking between alternatives A_3 and A_5 (now: $P_3 \geq P_5$ from (2)). Then, by following an approach similar to the previous one, we get that in order to reverse the current ranking of alternatives A_3 and A_5, the threshold value ${}_{3,4,5}$ should satisfy the following relation:

$${}_{3,4,5} < a_{34} \times \left[1 - \sqrt[W_4]{R\left(\frac{A_5}{A_3}\right)} \right].$$

Similarly with above, the following condition should also be satisfied for the new a'_{34} to have a feasible meaning:

$${}_{3,4,5} \leq a_{34}.$$

The above derivations can easily be expanded and are generalized in Theorem 5.

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